AIR FORCE INSTITUTE OF TECHNOLOGY

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FREQUENCY DEPENDENCE OF IONOSPHERIC ABSORPTION

THESIS

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FREQUENCY DEPENDENCE OF IONOSPHERIC ABSORPTION

THESIS

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the Air Force Institute of Technology

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in Partial Fulfillment of the

Requirements for the Degree of

Master of Science

by

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Graduate Space Physics

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Preface

What has been learned in the past few months will, undoubtedly, be reflected in the following pages; moreover, so will changing attitudes and points of view. Thus what was solely an absorption problem at the beginning has grown somewhat in the intervening time reflecting, perhaps, increasing awareness of the problems with which ionospheric physics is confronted, which, while being immediate in character, are always difficult to handle with the utmost preciseness, so that I have had to draw from other disciplines and, though I was not assured of more precisenss, these disciplines did help in a greater understanding of the problem and its related facts. In the paper, this is translated into certain basis features. First, a fairly lenghy examination of certain aspects of magneto-ionic theory has been included which, in my opinion, is essential to my understanding of the calculation of the absorption integrals. This examination forms the core of Chapters II and III. While I realize that it may be hard going for anybody reading this paper, its inclusion also reflects the amount of time that was spent on it. In Chapter IV, particularization is made to the more relevant expressions to the calculations themselves. The rest of the Chapters concern themselves with results, discussion and conclusions along with pertinent recommendations. Much material has been omitted which was related to the topic although not in a very direct way. This is reflected in the Bibliography; which contains all the relevant literature which I had occasion to refer to during the time of work. It is hoped that anybody wanting to pursue the subject further may use the Bibliography as a guide of where to look first.

During the time that this work was being done, several people cooperated in one way or another and to them goes my appreciation: to Mr. Richard S. Allen, who, with his unfailing enthusiasm and ever present optimism, made an optimist out of me, at least some of the time; to Mrs. Isabel Hussey, who helped in the computer input and output formats and some of the graphing; to Miss. Patricia Kennan, who did the programming, for her patience; and to Mr. Paul Lyons, who did some of the graphs quickly when there was not too much time. To all others, whose names I may omit, but were in some way instrumental to the final work, I also express my gratitude and, very especially, to all those who, despite my better judgement, convinced me not to write this paper in Spanish.

Julio Edgardo Barety

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Abstract

The magneto-ionic theory is examined in its relation to absorption calculations. For representative profiles the complex index of refraction was computed using the Appleton-Hartree and Sen-Wyller magneto-ionic formulas. The height profiles of the real and imaginary part of the index of refraction are plotted and the absorption corresponding to day-time and night-time conditions is computed and its frequency dependence determined. It is determined that absorption for general conditions also follows closely an inverse square law, although there are deviations, particularly for the extraordinary ray. It is determined that general propagation conditions do not affect radically the possible use of differential absorption for total absorption measurements. A possible method for doing this is given. The variation of the absorption coefficient as a function of parameter N is also determined and the linearity condition $\kappa = K(\gamma,\omega)N$ is shown to hold under certain conditions for general propagation conditions.

I. INTRODUCTION

In radio communications and atmospheric sciences in general, there has always been interest in determining the absorption of radio waves on traversal through the ionosphere. As early as 1936, Appleton (Ref 4) made calculations under simplified conditions. These simplified conditions included the quasi-longitudinal approximation, which is equivalent to propagation of the wave in a direction nearly parallel to the magnetic field; non-deviate absorption, i.e. the index of refraction M is nearly 1 and a Chapman-Layer model for the absorbing layer. At that time the Chapman-Layer model was considered to be valid for the E-layer. Later (1946), Jaeger (Ref 27) performed calculations of the same type extending Appleton's work to deviative absorption. He also considered reflections from the top side for a Chapman-Layer. The absorption for topside reflections will, in general, be different from that for reflections from the bottom-side because of the asymmetry of the Chapman-Layer around the height of maximum ionization. Jaeger retained the other restrictions that Appleton used. Another detail which should be noted is that both were based on the classical magneto-ionic formula for the index of refraction derived by Appleton (Ref 3).

The ionosphere, however, is much complicated than the use of some of the previous restrictions would seem to imply. In particular, the electron density profiles of the ionosphere are quite different

from the Chapman-layer model. Also the collision frequency profiles are more complicated than the simple exponential decrease used in the calculations mentioned in the first paragraph. The use of more realistic profiles leads to the use of electron density profiles and collision frequency profiles for which no analytical expression exists. This means that, in general, analytical integration of the absorption integrals is not possible. Finally, the quasi-longitudinal approximation, though quite good for a great range of conditions (Ref 45:79) is only a particular case of general propagation of a wave in a plasma medium upon which a constant magnetic field has been imposed.

The author has endeavored in this paper to calculate the absorption integrals for certain representative conditions as represented by typical electron density profiles and collision frequency profiles. The conditions imposed on the integral were fairly relaxed in the general direction indicated by the previous paragraph. They were computed for both magneto-ionic components of the wave.

The General Problem

A small problem such as this one usually forms part of a larger one. This larger problem has motivated the author in the work related to the present, more specific problem.

The general problem is the determination of the energy spectrum of cosmic radio sources (Ref 11), which may help determine the energy mechanisms by which they emit radio waves. It is observed that as

the frequency decreases from very high values towards the HF region, the spectral intensity increases. It may be assumed, however, that there should be a drop in intensity as the frequency approaches zero, in accordance with the physical laws that govern the behaviour of radiating sources. The shape of the spectral curves between 5 and 30 MC/S is very much in doubt at the moment. A knowledge of the shape of the energy curves in the 5-30 MC/S range could very well help determine what type of emission mechanism is present; whether thermal sources with a spectral response similar to that of a blackbody, or non-thermal sources like relativistic particles emitting radiation when trapped in magnetic fields, are responsible for the emission. The reason why the behaviour of the spectrum of the radio sources is in doubt between 5 and 30 MC/S is that it is masked by the day to day variations of strong absorption in the lower parts of the ionosphere. It is hoped that an understanding of the absorption undergone by a wave passing through the ionosphere will be a positive step in resolving this difficulty.

The fairly broad glimpse given here serves to emphasize the connection between the problem being examined in this paper and the broader one of cosmic radio sources.

The Mathematical and Physical Context

Apart from the manner in which the radio spectrum problem fits into the overall context there is also a mathematical and physical

context on which the present calculations are based. For the absorption calculations presented in this paper, it is necessary to know the refractive index of the medium and it can be calculated from the magneto-ionic theory. This theory forms the immediate mathematical and physical context of the present calculations. Therefore, the present paper devotes considerable space to those aspects of the theory which are essential for the absorption calculations. The physical considerations leading to the derivations of the useful equations have been examined at length. These derivations follow closely those given in the literature; not all has been included, however, and the interested reader is referred to the oroginal sources. Furthermore, this paper examines the theory on two different levels; the simple, original point of view by Appleton and termed "classical", and the more sophisticated approach of Sen and Wyller. As will be seen, these approaches give the index of refraction at a point as a function of the value of certain parameters at that point. The equations are then valid even for inhomogeneous media. To get the attenuation of the wave, a simplified treatment of Maxwell's equations for inhomogeneous media exists. It is called the WKB method and the solutions give an indication of the attenuation of the wave in a slowly-varying medium (Ref 12: Ch. 9). A few words have been set down on the WKB solutions at the end of Chapter 2.

Therefore, the next two chapters occupy themselves with classical magneto-ionic theory and its generalizations. The fourth chapter is

about numerical applications and specifically about the present calculations. The final chapter concerns itself with a discussion of results and some conclusions and recommendations.

II. CLASSICAL MAGNETO-IONIC THEORY

The magneto-ionic theory serves as the basis for most calculations in the ionosphere. It concerns itself with the propagation of electromagnetic waves through a magneto-ionic medium. A magneto-ionic medium is defined by Ratcliffe (Ref 45:8), as one in which free electrons and heavy positive ions are situated in a uniform magnetic field and are distributed with statistical uniformity, so that there is no resultant space charge. For most purposes this is sufficient, although sometimes it is necessary to include neutral particles and negative ions. To treat the absorption problem, it is possible to use two points of view: The macroscopic and the microscopic. In the former point of view an explanation of what happens to the elementary charges comprising the medium is avoided. Instead the parameters ϵ, the relative capacitivity; σ, the conductivity; and μ_m, the relative permeability are used to account for the effect on the electric field of equal numbers of oppositely charged particles and also of the translational motion of bound charges. σ, obviously, is related to the translational motion of the constituent charges. Since in many media these parameters can be measured with relative ease, the macroscopic point of view is the more useful. In plasmas some modifications are, however, necessary because in a plasma the constitutive parameters of the medium are not easily measured. The reason for this is that the space charges present and the boundary condition imposed by the measuring apparatus affect each other significantly. The

modification then is to use a microscopic approach to determine the constitutive parameters and then use these in Maxwell's equations to determine the fields. This approach is what mainly constitutes the Lorentz theory.

Classical Magneto-Ionic Theory

Brandstatter (Ref 10:40) gives a brief description of this theory. The Lorentz theory deals directly with the elementary charges of the medium. The theory dispenses with the concept of the medium. Instead, it visualizes the positive and negative charges which constitute the medium to be in free space. It postulates that all permeable bodies are composed of a large number of charged microscopic particles separated by free space. In conducting bodies, these microscopic particles are capable of being moved through the body because of electrical forces; that is, in non-conducting, or weakly conducting bodies, elastic forces are presumed to bind the particles to some equilibrium position. Any displacement that they may undergo from their position of equilibrium is very small, and when displacement occurs the medium becomes polarized. It is also assumed that the net charge of the medium is zero.

Furthermore, free displacement currents are assumed to exist also within the particles in addition to their existence in the free space between the particles. The way, then, that the effect of the medium is taken into account is to consider the motions of the charged particles in electromagnetic fields. Then whether the charges are

free or bound, a convection current can be defined. If in a certain portion of space there are N particles per unit volume, each with a charge q and a mass m, then if $\underline{v}$ is the time average of the charge velocity, each will produce a current $q\underline{v}$ and for the totality the current will be $Nq\underline{v}$. Hence, the average convection current $Nq\underline{v} = \rho\underline{v} = \underline{J}$.

It is not clear, however, whether $\underline{J}$ is proportional to the electric field or proportional to $\dot{\underline{E}}$, and out of phase with $\underline{E}$ by $\pi/2$ (i.e. whether it is a conduction current or a displacement current). So, in the most general case, it should be the sum of the free displacement current and the convection current.

Summarizing:

The Lorentz microscopic theory describes the electromagnetic phenomena by the same set of Maxwell's equation as the macroscopic theory:

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \tag{1.1}$$

$$\nabla \times \underline{H} = \underline{J} + \dot{\underline{D}} \tag{1.2}$$

$$\nabla \cdot \underline{D} = \rho \tag{1.3}$$

$$\nabla \cdot \underline{B} = 0 \tag{1.4}$$

but since there is no medium in this point of view, the equations:

$$\underline{B} = \mu_m \underline{H} \tag{1.5}$$

$$\underline{D} = \epsilon \underline{E} \tag{1.6}$$

$$\underline{J} = \sigma \underline{E} \tag{1.7}$$

are given "in vacuo",

$$\mu_m = \mu_o \quad ; \quad \epsilon = \epsilon_o \tag{1.8}$$

"and the current density is given directly in terms of the moving charges:

$$\underline{J} = \sum q_\kappa v_\kappa / \text{ volume of region containing charges.} \tag{1.9}$$

Maxwell's equations relate the field to the charges and their motions, but since these are not known, it is necessary to supplement them with the dynamical equations of motion for the charges. These equations have the form, for each particle, q_κ :

$$\frac{d}{dt}(m_\kappa v_\kappa) = q_\kappa (\underline{E} + \underline{v}_\kappa \times \underline{B}) \tag{1.10}$$

where $\underline{B}$ is the <u>total</u> magnetic induction field evaluated at the position $\underline{r}_\kappa$, of the κ^{th} particle; $\underline{E}$ is the total electric field evaluated at $\underline{r}_\kappa$. These fields can consist in part of external fields, and in part of fields due to the q_κ themselves.

The set of equations must be solved simultaneously for the dynamical behavior of the charges, and from this behavior we can

derive the constitutive parameters for the microscopic model of the medium.

In the Lorentz theory, the motion of the electrical charges is described in terms of the polarization vector (dipole moment per unit volume) rather than in terms of the convection current. In this case the polarization vector is $\underline{P} = N\underline{r}$, $\underline{r}$ being the average displacement of the charged particles. Consequently, $\underline{J} = \dot{\underline{P}}$.

In the present absorption calculations it is necessary to know how the index of refraction varies with some of the parameters of the ionosphere. It is the magneto-ionic theory which provides the answer.

A. - THE WAVE EQUATION

In a magneto-ionic medium as defined by Ratcliffe, the net free charge $\rho = 0$. Maxwell's equations then became:

$$\begin{aligned} \nabla \cdot \underline{P} &= 0 \\ \nabla \cdot \underline{B} &= 0 \\ \nabla \times \underline{E} &= -\frac{\partial \underline{B}}{\partial t} \\ \nabla \times \underline{H} &= \underline{J} + \frac{\partial \underline{P}}{\partial t} \end{aligned} \qquad (1.3)$$

It is convenient to define an effective displacement $\underline{D}_e$

$$\frac{\partial \underline{D}_e}{\partial t} = \underline{J} + \frac{\partial \underline{\nu}}{\partial t}$$

furthermore, all field components in Maxwell's equations are assumed to contain the factor

$$e^{i(\omega t - \underline{\kappa}\cdot\underline{r})}$$

A new notation is introduced to simplify certain calculations. Vectors will be written in their component form, so that an equation of the form $\vec{V}_i = \frac{\partial}{\partial x_i}$ will be taken to mean the vector ∇ has components $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$ in three dimensions. Furthermore by introducing the quantity ϵ_{ijk} defined as follows:

$$\epsilon_{ijk} = 0 \tag{1.10}$$

if i, j, k are not all distinct,

$$\epsilon_{ijk} = 1$$

When the order of i, j, k, is the result of an even permutation on $1, 2, 3$, and

$$\epsilon_{ijk} = -1$$

When the order of i, j, k, is the result of an odd permutation on $1, 2, 3$, the cross product of two vectors can be written in compressed form. For example:

$$(\underline{A} \times \underline{B})_i = \epsilon_{ijk} A_j B_k \tag{1.11}$$

Where the summation convection has been used for the indices j and k. As a check, in three dimensions, the x-component of $\underline{A} \times \underline{B}$ can be expanded:

$$(\underline{A} \times \underline{B})_1 = \epsilon_{1jk} A_j B_k = \epsilon_{11k} A_1 B_k + \epsilon_{12k} A_2 B_k + \epsilon_{13k} A_3 B_k$$

In the first sum all terms will be zero since ϵ_{11K} already has repeated indices; in the second sum the only non-zero term will occur when K=3, i.e. $\epsilon_{123} A_2 B_3 = (+1) A_2 B_3$; and in the third sum the only non-zero term will occur when K=2, i.e. $\epsilon_{132} A_3 B_2 = (-1) A_3 B_2$

Therefore

$$(\underline{A} \times \underline{B})_1 = (A_2 B_3 - A_3 B_2) \tag{1.12}$$

This can be seen to have the typical form of the components of a cross-product.

Therefore, if $A_K = F_K e^{i(\omega t - \underline{K} \cdot \underline{r})}$, where E is independent of position, then

$$(\nabla \times A)_i = \epsilon_{ijK} \frac{\partial}{\partial x_j} F_K e^{i(\omega t - K_j x_j)}$$

$$= -\epsilon_{ijK} F_K e^{i(\omega t - K_j x_j)} (iK_j)$$

$$= -i (\underline{K} \times \underline{A})_i \tag{1.13}$$

Then

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \tag{1.14}$$

becomes

$$-i \underline{K} \times \underline{E} = -i\omega \underline{B} \tag{1.15}$$

and $\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}$,

$$-i \underline{K} \times \underline{H} = i\omega \underline{D}_e \tag{1.16}$$

Now, using the relations between the fields given "in vacuo", equation (1.15) becomes:

$$\underline{\kappa} \times \underline{E} = \omega \mu_o \underline{H} \tag{1.17}$$

taking the cross-product with $\underline{\kappa}$,

$$\underline{\kappa} \times (\underline{\kappa} \times \underline{E}) = \omega \mu_o \underline{\kappa} \times \underline{H} \tag{1.118}$$

Using the vector identity $\underline{A} \times (\underline{B} \times \underline{C}) = (\underline{A} \cdot \underline{C})\underline{B} - (\underline{A} \cdot \underline{B})\underline{C}$ on the left and equation (1.16) on the right,

$$(\underline{\kappa} \cdot \underline{E})\underline{\kappa} - (\underline{\kappa} \cdot \underline{\kappa})\underline{E} = -\omega^2 \mu_o \underline{D_e} \tag{1.19}$$

and, finally, from

$$\frac{\partial \underline{D_e}}{\partial t} = i\omega \underline{D_e} = \underline{J} + \frac{\partial \underline{D}}{\partial t} = \underline{J} + i\omega \underline{D} \tag{1.20}$$

$$\underline{D_e} = \frac{\underline{J}}{i\omega} + \underline{D} \tag{1.21}$$

$$\underline{D_e} = \frac{\sigma \underline{E}}{i\omega} + \epsilon_o \underline{E} \tag{1.22}$$

$$\underline{D_e} = \epsilon_o \left(\frac{\sigma}{i\omega \epsilon_o} + \underline{I}\right)\underline{E} \tag{1.23}$$

where $\underline{I}$ is the identify matrix.

In analogy with dielectric materials, this equation is written as

$$\underline{D_e} = \epsilon_o \epsilon_e \underline{E} \tag{1.24}$$

where ϵ_e is called the effective dielectric tensor of the plasma.

Equation (1.3) then becomes:

$$(\underline{K}\cdot\underline{E})\underline{K} - (\underline{K}\,\underline{K})\underline{E} = -\omega^2\mu_0\epsilon_0\epsilon_e\underline{E} \tag{1.25}$$

This is the wave equation used in the calculation of the complex refractive index. What is needed for this calculation is the assumption of a form for $\underline{K}$, the wave vector, and the calculation of the dielectric tensor ϵ_e. This is where the equations of motion play a part. It is at this point that one is really justified in saying that a generalization of the dielectric tensor (or conductivity) implies a generalization of the Appleton-Hartree formula.

B - <u>THE EQUATIONS OF MOTION</u>

The equations of motion as already stated, are:

$$\frac{d}{dt}(m_\kappa \underline{V}_\kappa) = q_\kappa(\underline{E} + \underline{V}_\kappa \times \underline{B}) \tag{1.10}$$

but these, however, do not take into account forces which may be introduced by collisional processes. For example, neglecting the magnetic field for simplicity, for an electron, (1.10) becomes:

$$m\dot{\underline{v}} = e\underline{E}_0 e^{i\omega t} \tag{1.26}$$

where $\underline{E}_0$, may depend on position. This equation is easily integrated

$$\underline{v} = -i\left(\frac{e\underline{E}_0}{m\omega}\right)e^{i\omega t} + \underline{C}, \tag{1.27}$$

This velocity, it must be remembered, is imposed by the electric field and must be distinguished from random velocity. After a collision this "ordered" velocity impressed by the field must be zero, so at time t_1, when the last collision took place, $\underline{v} = 0$ and

$$\underline{c}_1 = i\left(\frac{e\underline{E}_0}{m\omega}\right)e^{i\omega t_1} \tag{1.28}$$

therefore,

$$\underline{v} = +i\left(\frac{e\underline{E}_0}{m\omega}\right)\left(e^{i\omega t_1} - e^{i\omega t}\right) \tag{1.29}$$

If t is set equal to $t_1 + \tau$, equation (1.29) can be written as

$$\underline{v} = -i\left(\frac{e\underline{E}_0}{m\omega}\right)e^{i\omega t}\left(1 - e^{-i\omega\tau}\right) \tag{1.30}$$

In a sense equation (1.30) gives the ordered velocity of an electron in terms of the time of its last collision. If the number of collisions in a time dt were known the average velocity of the electron could be calculated. For ease in future calculations, an equation of motion could be constructed such that the calculated average velocity of the electron would be its solution. This equation would include an expression for a "force" due to collisions. The times between collisions are supposed to have a Poisson distribution f, i.e.

$$f = N\nu e^{-\tau\nu}\,d\tau \tag{1.31}$$

where ν is the collision frequency, then the average ordered velocity is

$$\langle \underline{v} \rangle = \frac{1}{N}\int_0^\infty \left(\frac{-ie\underline{E}_0}{m\omega}\right) e^{i\omega t}\left(1 - e^{-i\omega\tau}\right) N\nu e^{-\tau\nu}\, d\tau \qquad (1.32)$$

$$= \left(-\frac{i\nu e\underline{E}_0}{m\omega}\right)\int_0^\infty e^{i\omega t}\left(1 - e^{-i\omega\tau}\right) e^{-\tau\nu}\, d\tau$$

$$= \frac{-ie\nu\underline{E}_0}{m\omega} e^{i\omega t}\int_0^\infty \left(e^{-\tau\nu} - e^{-(i\omega+\nu)\tau}\right) d\tau$$

$$= -\frac{ie\nu\underline{E}_0}{m\omega} e^{i\omega t}\left[-\frac{e^{-\tau\nu}}{\nu}\Big|_0^\infty + \right. \qquad (1.33)$$

$$\left. \frac{e^{-(i\omega+\nu)\tau}}{i\omega+\nu}\Big|_0^\infty\right]$$

$$= -\frac{ie\nu E_0}{m\omega} e^{i\omega t}\left[\frac{1}{\nu} - \frac{1}{i\omega+\nu}\right]$$

$$= -\frac{ie\nu E_0}{m\omega} e^{i\omega t}\left[\frac{i\omega+\nu-\nu}{\nu(i\omega+\nu)}\right]$$

$$\langle v\rangle = \frac{eE_0}{m} e^{i\omega t}\left(\frac{1}{i\omega+\nu}\right) \qquad (1.34)$$

If the effect of collisions was to be accounted for with a force $-g\underline{v}$ where g is a constant, then equation (1.26) would be transformed into:

$$m\dot{\underline{v}} + g\underline{v} = e\underline{E}_0 e^{i\omega t} \qquad (1.35)$$

Its solution is:

$$\underline{v} = \frac{e\underline{E}_0}{(i\omega m + g)} e^{i\omega t} \qquad (1.36)$$

If this solution is to represent the average velocity of the electron as explained before, comparison between equation (1.36) and equation (1.34) leads to

$$g = m\nu \tag{1.37}$$

so that the fictional force produced by the collisions is $-m_i \nu \underline{v}$.

The analysis here has been simplified but a more detailed analysis will change the term by at most a multiplicative constant.

Written in full then the equations of motion are:

$$\frac{d}{dt}(m_\kappa \underline{v}_\kappa) = -m_\kappa \nu \underline{v}_\kappa + q_\kappa(\underline{E} + \underline{v}_\kappa \times \underline{B}) \tag{1.38}$$

C - SOLUTION OF THE EQUATIONS OF MOTION

The subscripts will be dropped. It is understood that there are N similar equations; one for each electron. Furthermore, all quantities are considered to be time-harmonic, i.e. they contain the factor $e^{i\omega t}$. Therefore, equation (1.38) becomes:

$$-m\omega^2 \underline{r} = -i\omega m \nu \underline{r} + e(\underline{E} + i\omega \underline{r} \times \underline{B}) \tag{1.38}$$

The magnetic field $\underline{B}$ as previously stated was composed of the steady external field and that of the wave but it can be shown that the magnetic field of the wave has a very small effect on the electrons (Ref 12:28) (See Appendix A), therefore, it is neglected.

Setting $\underline{B} = \underline{B}_0$ (the external steady magnetic field), equation (1.38) becomes:

$$-m\omega^2 \underline{r} = -i\omega m\nu \underline{r} + e(\underline{E} + i\omega \underline{r} \times \underline{B}_0) \tag{1.39}$$

Some simplification is possible if an appropriate coordinate system is chosen. The one chosen here has its Oz axis parallel to $\underline{B}_0$ (see Fig 1). After some rearrangement equation (1.39) becomes:

$$-\left(\frac{m\omega^2}{e}\right)\left(\beta \underline{r} - \frac{ie}{m\omega} \underline{B}_0 \times \underline{r}\right) = \underline{E} \tag{1.40}$$

where $\beta = 1 - i\frac{\nu}{\omega}$, because

$$\frac{e}{m\omega} (\underline{B}_0 \times \underline{r})_i = \frac{e}{m\omega} \epsilon_{ijk} B_j r_k \tag{1.38}$$

and since $B_3 \neq 0$ while $B_1 = B_2 = 0$, as can be seen by referring to Figure (1),

$$\frac{e}{m\omega} \epsilon_{ijk} B_j r_k = \frac{e}{m\omega} \epsilon_{i3k} B_3 r_k \tag{1.41}$$

The right hand side can be thought of as a matrix multiplication between a matrix whose elements are given by ϵ_{i3k} and a column vector $(r_k) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Using the definition of ϵ_{i3k}, the elements of this martrix can be ascertained to be $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

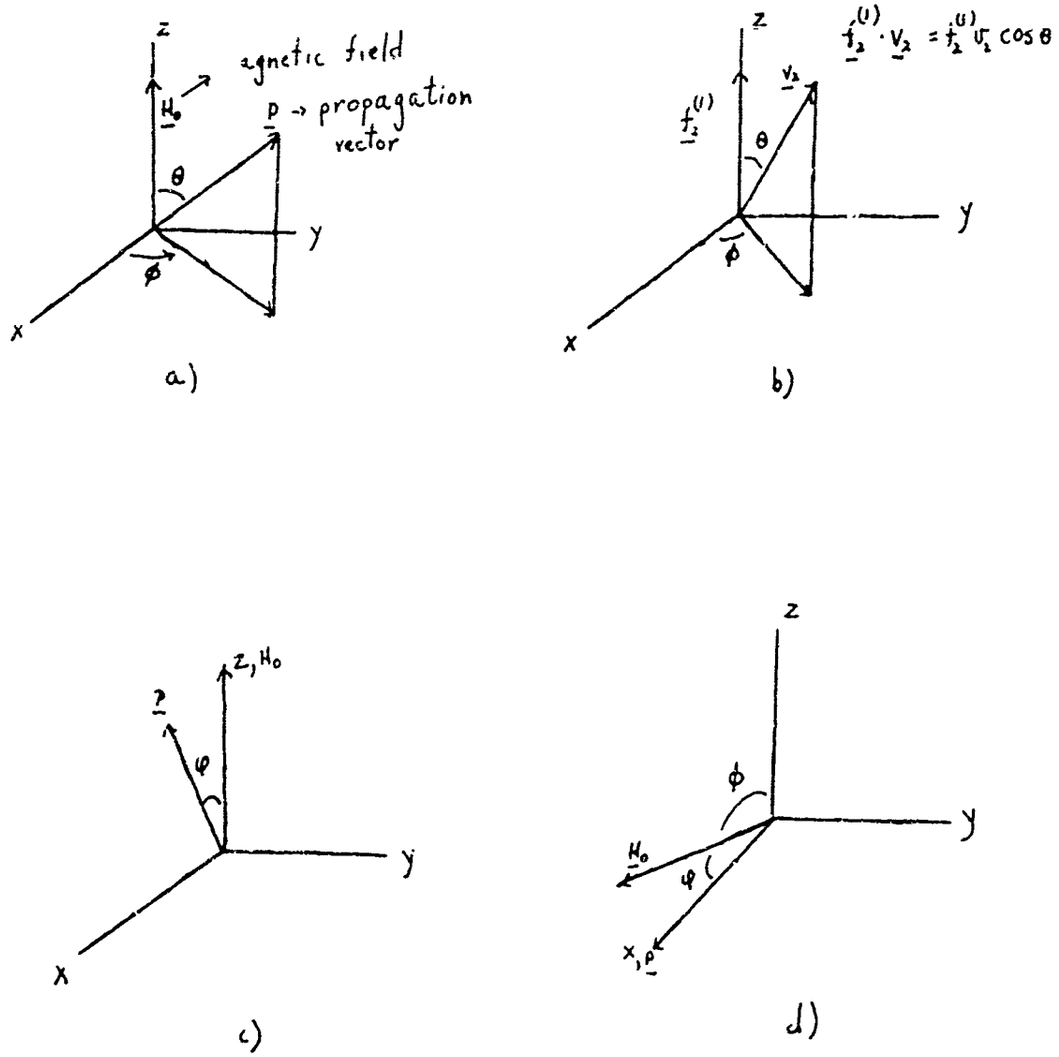


Fig. 1
Some coordinate systems used in the derivation of the Appleton-Hartree magneto-ionic formula.

c) and d) Ref (49)

Therefore equation (1.38) can be written in matrix form as follows:

$$\frac{eB_0}{m\omega}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \frac{e}{m\omega}\left[(\underline{B}_0 \times \underline{r})\right] \qquad (1.42)$$

At this point several definitions are useful to make some economy in writing. First $\frac{eB_0}{m} = s$, the gyrofrequency of the electron, and secondly $\frac{s}{\omega} = \Omega$. Furthermore,

$$\frac{-eB_0}{m\omega}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = \hat{\Omega} \qquad (1.43)$$

with these definitions in mind, equation (1.42) becomes:

$$\frac{e}{m\omega}\left[\underline{B}_0 \times \underline{r}\right] = -\hat{\Omega}\,\underline{r} \qquad (1.44)$$

If equation (1.44) is used, equation (1.40) can be compressed further:

$$-\frac{m\omega^2}{e}(\beta I + i\hat{\Omega})\,\underline{r} = \underline{E} \qquad (1.45)$$

where I is the identity matrix.

It is understood that this equation is a matrix equation. It can be seen easily that $(\beta I + i\hat{\Omega})$ is non-singular, therefore, it has an inverse, and the solution to the equations of motion is

$$\underline{r} = -\frac{e}{m\omega^2}(\beta I + i\hat{\Omega})^{-1}\underline{E} \qquad (1.46)$$

D - THE DIELECTRIC TENSOR

The solution to the equations of motion just obtained is not very illustrative in itself. Its main usefulness lies in the fact that from them the conductivity and dielectric tensor can be calculated. It will be recalled that this is the problem which the Lorentz theory promised to solve, and in turn the determination of these constitutive parameters makes it possible to determine the index of refraction of the medium from the wave equation (1.25).

It will be recalled, that

$$\underline{J} = \hat{\sigma}\,\underline{E} \tag{1.7}$$

and also, that

$$\underline{J} = \dot{\underline{P}} \tag{1.47}$$

Since all field quantities are assumed to be time harmonic, i.e. they contain the factor $e^{i\omega t}$, then

$$\dot{\underline{P}} = i\omega \underline{P} \tag{1.48}$$

Furthermore, since $\underline{P} = Ne\underline{r}$, then $\dot{\underline{P}} = i\omega Ne\underline{r}$.

From the solution to the equations of motion, it is clear that

$$\underline{J} = i\omega Ne\left(\frac{-e}{m\omega^2}\right)\left(\beta I + i\hat{\Omega}\right)^{-1}\underline{E}$$

$$\underline{J} = -\epsilon_0 i\omega\left(\frac{\omega_e^2}{\omega^2}\right)\left(\beta I + i\hat{\Omega}\right)^{-1}\underline{E} \tag{1.49}$$

Where $\omega_e^2 = \frac{Ne^2}{m\epsilon_0}$ is the square of the plasma frequency. It is clear from equation (1.49) that the conductivity tensor is given by

$$\hat{\sigma} = -i\epsilon_0\omega\left(\frac{\omega_0^2}{\omega^2}\right)(\beta I + i\hat{\Omega})^{-1} \tag{1.50}$$

Since the dielectric tensor is defined by

$$\frac{\hat{\sigma}}{i\omega\epsilon_0} + I \tag{1.51}$$

it is clear then that

$$\hat{\epsilon}_e = -a(\beta I + i\hat{\Omega})^{-1} + I \tag{1.52}$$

where $a = \frac{\omega_0^2}{\omega^2}$

When $-a(\beta I + i\hat{\Omega})^{-1}$ is set equal to S, then

$$\hat{\epsilon}_e = S + I \tag{1.53}$$

S is called the susceptibility matrix.

At this stage one could proceed to calculate a dispersion relation from the wave equation (1.25) and therefore get the index of refraction of the medium. It is, however, preferable to have $\hat{\epsilon}_e$ in diagonal form; that would certainly simplify the dispersion relation, which, as a rule, is always fairly complicated. This latter procedure is done by every author consulted. Briefly: it consists of finding a non-singular matrix A such that

$$A^{-1}\hat{\epsilon}_e A = \bar{\epsilon}_e \tag{1.54}$$

where $\tilde{\epsilon}_c$ is diagonal. This is called a similarity transformation and $\tilde{\epsilon}_c$ is said to be similar to $\hat{\epsilon}_c$. From equation (1.53) it can be seen that the same matrix A which diagonalizes $\hat{\epsilon}_c$, also diagonalizes S and hence $\hat{\sigma}$ since $S = \frac{\hat{\sigma}}{i\omega\epsilon_0}$. By the same token if it diagonalizes $\tilde{\sigma}$ it diagonalizes $(\beta I + i\hat{\Omega})^{-1}$. And since

$$(\beta I + i\hat{\Omega})^{-1}(\beta I + i\hat{\Omega}) = I \tag{1.55}$$

$$A^{-1}(\beta I + i\hat{\Omega})^{-1} A A^{-1}(\beta I + i\hat{\Omega}) = A^{-1} I \tag{1.56}$$

$$D A^{-1}(\beta I + i\hat{\Omega}) = I A^{-1} \tag{1.57}$$

where D is a diagonal matrix by hypothesis. If both sides of equation (1.57) are multiplied by A from the right, equation (1.57) becomes

$$D A^{-1}(\beta I + i\hat{\Omega}) A = I \tag{1.58}$$

If it is further supposed that the elements in the diagonal of D are all non-zero, then D is invertible, and hence

$$A^{-1}(\beta I + i\hat{\Omega}) A = D^{-1} \tag{1.59}$$

But the inverse of a diagonal matrix is also diagonal, hence it has been proved that A also diagonalizes $(\beta I + i\hat{\Omega})$.

There is a theorem in matrix algebra which asserts that if the eigenvalues of a matrix are all distinct, then the diagonal matrix with the eigenvalues along the main diagonal is similar to the given

matrix. Hence if the eigenvalues of ($\beta I + i\hat{\Omega}$) are determined, and if they turn out to be distinct, D^{-1} can be written down automatically.

The characteristic equation of ($\beta I + i\hat{\Omega}$) is:

$$\begin{vmatrix} \beta-\lambda & i\Omega & 0 \\ -i\Omega & \beta-\lambda & 0 \\ 0 & 0 & \beta-\lambda \end{vmatrix} = 0 \qquad (1.60)$$

or

$$(\beta-\lambda)\left[(\beta-\lambda)^2 - \Omega^2\right] = 0 \qquad (1.61)$$

from which the eigenvalues are seen to be:

$$\lambda_1 = \beta+\Omega \qquad (1.62)$$

$$\lambda_2 = \beta-\Omega \qquad (1.63)$$

$$\lambda_3 = \beta \qquad (1.64)$$

Therefore,

$$D^{-1} = \begin{pmatrix} \beta+\Omega & 0 & 0 \\ 0 & \beta-\Omega & 0 \\ 0 & 0 & \beta \end{pmatrix} \qquad (1.65)$$

since all the eigenvalues are distinct.

From equation (1.59)

$$(\beta I + i\hat{\Omega})A = AD^{-1} \qquad (1.66)$$

is obtained. Since

$$\beta \underline{I} + i\hat{\Omega} = \begin{pmatrix} \beta & i\Omega & 0 \\ -i\Omega & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix}$$

and D^{-1} is as given by (1.65), then the relations between the elements of A are:

$$\begin{aligned}
\beta A_{11} + i\Omega A_{21} &= (\beta+\Omega) A_{11} \\
-i\Omega A_{11} + \beta A_{21} &= (\beta+\Omega) A_{21} \\
\beta A_{31} &= (\beta+\Omega) A_{31} \\
\beta A_{12} + i\Omega A_{22} &= (\beta-\Omega) A_{12} \\
-i\Omega A_{12} + \beta A_{22} &= (\beta-\Omega) A_{22} \\
\beta A_{32} &= (\beta-\Omega) A_{32} \\
\beta A_{13} + i\Omega A_{23} &= \beta A_{13} \\
-i\Omega A_{13} + \beta A_{23} &= \beta A_{23} \\
\beta A_{33} &= \beta A_{33}
\end{aligned} \qquad (1.67)$$

Not all equations in this group are independent, so that not all the elements of A can be determined. This merely means that the matrix

A, is not unique. The relations that can be obtained are the following:

$$\frac{A_{11}}{A_{21}} = +i$$

$$A_{31} = 0$$

$$\frac{A_{12}}{A_{22}} = -i$$

$$A_{32} = 0 \qquad (1.68)$$

$$A_{23} = 0$$

$$A_{13} = 0$$

$$A_{33} = \text{Arbitrary}$$

$$(a + bi)(a - bi)` = a^2 + b^2 \qquad (1.69)$$

The elements in columns of the matrix A are frequently interpreted to be eigenvectors. Their magnitude should be 1. Therefore:

$$|A_{11}|^2 + |A_{21}|^2 + |A_{31}|^2 = 1 \qquad (1.70)$$

$$\left|\frac{A_{11}}{A_{21}}\right|^2 = \frac{1}{|A_{21}|^2} - 1 \qquad (1.71)$$

$$2 = \frac{1}{|A_{21}|^2} \tag{1.72}$$

$$\therefore \quad |A_{21}| = \frac{1}{\sqrt{2}} \tag{1.73}$$

And also

$$|A_{11}| = \frac{1}{\sqrt{2}} \tag{1.74}$$

Similar equations hold for the second column, and for the third $A_{33} = 1$. Thus, as a matter of convenience, the matrix here chosen is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ -i & i & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \tag{1.75}$$

The inverse of this matrix A can be calculated in a straightforward manner and turns out to be:

$$A^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 \\ 1 & -i & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \tag{1.76}$$

It will be observed that this matrix is also the conjugate transpose of A denoted by A^{*1}. In other words, the matrix A is unitary.

The intent of these preliminaries was to obtain A so that the diagonalization of $\hat{\epsilon}_c$ could be effected, and thus simplify the final dispersion relation.

It is recalled that

$$\hat{\epsilon}_c = S + \underline{I} \tag{1.53}$$

since $S = \frac{\hat{\sigma}}{i\omega\epsilon_0} = -a(\beta\underline{I} + i\hat{\Omega})^{-1}$, and since A diagonalizes $(\beta\underline{I} + i\Omega)^{-1}$ and $(\beta\underline{I} + i\Omega)$, also diagonalizes S^{-1}, S^{-1} can be written as:

$$S^{-1} = -\frac{1}{a}(\beta\underline{I} + i\Omega)$$

Since $(\beta\underline{I} + i\Omega)$ has already been diagonalized, it is easy to see that:

$$A^{-1}S^{-1}A = -\frac{1}{a}\begin{pmatrix} \beta+\Omega & 0 & 0 \\ 0 & \beta-\Omega & 0 \\ 0 & 0 & \beta \end{pmatrix} \tag{1.77}$$

Therefore, from equation (1.53)

$$\bar{\epsilon}_c = A^{-1}\hat{\epsilon}_c A = -a\begin{pmatrix} \frac{1}{\beta+\Omega} & 0 & 0 \\ 0 & \frac{1}{\beta-\Omega} & 0 \\ 0 & 0 & 1/\beta \end{pmatrix} + \tag{1.78}$$

$$+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.78}$$

$$\bar{\epsilon}_e = \begin{pmatrix} 1 - \frac{a}{\beta+\Omega} & 0 & 0 \\ 0 & 1 - \frac{a}{\beta-\Omega} & 0 \\ 0 & 0 & 1 - \frac{a}{\beta} \end{pmatrix} \tag{1.79}$$

To reduce the writing, the following definitions are made:

$$\epsilon_1 = 1 - \frac{a}{\beta+\Omega} \tag{1.80}$$

$$\epsilon_2 = 1 - \frac{a}{\beta-\Omega} \tag{1.81}$$

$$\epsilon_3 = 1 - \frac{a}{\beta} \tag{1.82}$$

so that

$$\bar{\epsilon}_e = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \tag{1.83}$$

Finally, to transform the wave equation to this new system, where $\bar{\epsilon}_e$ is diagonal, the transformation law for the vectors must be

obtained. This may be obtained from equation (1.24).

$$\underline{D}_e = \epsilon_o \hat{\epsilon}_e \underline{E} \tag{1.24}$$

If the equation is multiplied by A^{-1} from the left, then

$$A^{-1}\underline{D}_e = \epsilon_o A^{-1}\hat{\epsilon}_e \underline{E} = \epsilon_o (A^{-1}\hat{\epsilon}_e A)A^{-1}\underline{E} = \epsilon_o \bar{\epsilon}_e A^{-1}\underline{E} \tag{1.84}$$

or

$$\bar{\underline{D}}_e = \epsilon_o \bar{\epsilon}_e \bar{\underline{E}} \tag{1.85}$$

where $\bar{D}e$, $\bar{E}$ denote $\underline{D}e$, $\underline{E}$, in the new system; therefore the transformation law for a vector $\underline{X}$

$$\bar{X} = A^{-1}\underline{X} \tag{1.86}$$

Where the bar on top denotes the vector in the new system. In the present calculations, equation (1.86) represents the following three equations:

$$(\bar{X})_1 = \frac{1}{\sqrt{2}}\left[(\underline{X})_1 + i(\underline{X})_2\right] \tag{1.87}$$

$$(\bar{X})_2 = \frac{i}{\sqrt{2}}\left[(\underline{X})_1 - i(\underline{X})_2\right] \tag{1.88}$$

$$(\bar{X})_3 = (\underline{X})_3 \tag{1.89}$$

These transformation equations are of great importance in the calculation of fields in a plasma.

The final calculations on the wave equation can now be made. If in the wave equation (1.25) the wave vector $\underline{K}$ is taken as

$$\frac{\omega}{V}\underline{n} \tag{1.90}$$

where V is the speed of the wave and $\underline{n}$ is a unit vector in the direction of propagation, the equation becomes

$$\left(\frac{\omega}{V}\right)^2(\underline{n}\cdot\underline{E})\underline{n} - \left(\frac{\omega}{V}\right)^2(\underline{n}\cdot\underline{n})\underline{E} = -\mu_0\epsilon_0\omega^2\hat{\epsilon}_e\underline{E} \tag{1.91}$$

$$(\underline{n}\cdot\underline{E})\underline{n} - (\underline{n}\cdot\underline{n})\underline{E} = -\mu_0\epsilon_0 V^2\hat{\epsilon}_e\underline{E} = -\mu_0 V^2\underline{D}_e \tag{1.92}$$

If this equation is transformed by A

$$(\underline{n}\cdot\underline{E})\bar{n} - \bar{E} = -\mu_0 V^2\bar{D}_e \tag{1.93}$$

where it has been kept in mind that the inner product of two vectors is invariant under linear, homogeneous transformations. The second term on the left of (1.93) can be written

$$(\bar{E})_i = \frac{(\bar{D}_e)_i}{\epsilon_0(\epsilon_e)_i} \tag{1.94}$$

where i denotes a particular component of the vectors. Therefore, in component form (1.93) can be written as

$$\frac{\bar{n}_i}{\frac{1}{\epsilon_0\bar{\epsilon}_i} - \mu_0 V^2} = \frac{(\bar{D}_e)_i}{(\underline{n}\cdot\underline{E})} \tag{1.95}$$

In general $(\underline{n} \cdot \underline{E})$ is not zero, thus avoiding the trivial results which would otherwise be obtained.

If the inner product of the terms in equation (1.16) with $\underline{n}$ is taken (i.e. with $\underline{\kappa}$) then:

$$-\underline{\kappa} \cdot (\underline{\kappa} \times \underline{H}) = \omega \underline{\kappa} \cdot \underline{D}_e \tag{1.96}$$

$$0 = \omega \underline{\kappa} \cdot \underline{D}_e \tag{1.97}$$

So that in this case $\underline{n}$ is orthogonal to $\underline{D}_e$. Assuming that $\underline{n}$ is real, then $\underline{n}^*$ is also orthogonal, and finally because a unitary transformation preserves the complex inner product, it is also true that:

$$\underline{n}^* \cdot \underline{D}_e = \bar{n}^* \cdot \underline{D}_e = 0 \tag{1.98}$$

where $\bar{n}^*$ is the transformed vector.

Therefore multiplying (1.95) by $\bar{n}^*$, the result is:

$$\frac{\bar{n}_1^* \bar{H}_1}{\frac{1}{\epsilon_0 \epsilon_1} - \mu_0 V^2} + \frac{\bar{n}_2^* \bar{n}_2}{\frac{1}{\epsilon_0 \epsilon_2} - \mu_0 V^2} + \frac{\bar{n}_3^* \bar{H}_3}{\frac{1}{\epsilon_0 \epsilon_3} - \mu_0 V^2} = 0 \tag{1.99}$$

or using equations (1.87), (1.88), (1.89):

$$\frac{\frac{1}{\sqrt{2}}(n_1 - i n_2)(n_1 + i n_2)}{\frac{1}{\epsilon_0 \epsilon_1} - \mu_0 V^2} + \frac{\frac{1}{\sqrt{2}}(n_1 + i n_2)(n_1 - i n_2)}{\frac{1}{\epsilon_0 \epsilon_2} - \mu_0 V^2} + \frac{n_3^2}{\frac{1}{\epsilon_0 \epsilon_3} - \mu_0 V^2} = 0 \tag{1.100}$$

$$\frac{\frac{1}{2}(n_1^2+n_2^2)}{\frac{1}{\epsilon_0\mu_0\epsilon_1}-V^2}+\frac{\frac{1}{2}(n_1^2+n_2^2)}{\frac{1}{\epsilon_0\mu_0\epsilon_2}-V^2}+\frac{n_3^2}{\frac{1}{\epsilon_0\mu_0\epsilon_3}-V^2}=0 \qquad (1.101)$$

and defining

$$\bar{V}_1^2=\frac{1}{\epsilon_0\mu_0\epsilon_1} \qquad (1.102)$$

$$\bar{V}_2^2=\frac{1}{\epsilon_0\mu_0\epsilon_2} \qquad (1.103)$$

$$\bar{V}_3^2=\frac{1}{\epsilon_0\mu_0\epsilon_3} \qquad (1.104)$$

the equation becomes:

$$\frac{\frac{1}{2}(n_1^2+n_2^2)}{\bar{V}_1^2-V^2}+\frac{\frac{1}{2}(n_1^2+n_2^2)}{\bar{V}_2^2-V^2}+\frac{n_3^2}{\bar{V}_3^2-V^2}=0 \qquad (1.105)$$

This is the dispersion equation which gives a quadratic equation for V^2. This means that under the conditions assumed in this derivation there are in general two waves which propagate at different velocities. These waves are called the ordinary and the extraordinary depending on how much they are affected by the magnetic field. These two modes of propagation are due to the effect of magnetic field. It will be seen that in the final formula if the magnetic field is set equal to zero, any distinctions between the two waves vanish.

D - THE APPLETON-HARTREE EQUATION

Supposing now that the direction of propagation is entirely arbitrary (Fig 1) then $\underline{n}$ can be written as

$$\underline{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta) \qquad (1.106)$$

in a spherical coordinate system; one can see that $n_1^2 + n_2^2 = \sin^2\theta$ and $n_3^2 = \cos^2\theta$ so that it is evident from (1.105) that the equation resulting from (1.105) is independent of ϕ, but not of θ, which, if it is recalled that the direction was chosen parallel to the magnetic field, is the angle between the direction of propagation and the magnetic field.

Equation (1.105) can, therefore, be written as:

$$\frac{\frac{1}{2}(\sin^2\theta)}{\bar{V}_1^2 - V^2} + \frac{\frac{1}{2}(\sin^2\theta)}{\bar{V}_2^2 - V^2} + \frac{\cos^2\theta}{\bar{V}_3^2 - V^2} = 0 \qquad (1.107)$$

Furthermore, the complex index of refraction M is equal to $\frac{V_0}{V}$ where V_0 is some reference velocity like the velocity of light, so that

$$V^2 = \frac{V_0^2}{M^2} \qquad (1.108)$$

and, equation (1.107) can be written as:

$$\frac{\frac{\sin^2\theta}{2}}{\bar{V}_1^2 - \frac{V_0^2}{M^2}} + \frac{\frac{\sin^2\theta}{2}}{\bar{V}_2^2 - \frac{V_0^2}{M^2}} + \frac{\cos^2\theta}{\bar{V}_3^2 - \frac{V_0^2}{M^2}} = 0 \qquad (1.109)$$

Now since

$$\bar{V}_i^2 = \frac{1}{\epsilon_0 \mu_0 \bar{\epsilon}_i} = \frac{c^2}{\bar{\epsilon}_i} \qquad (1.110)$$

and if $V_o^2 = c^2$ equation (109) can then be written

$$(18) \qquad \frac{\sin^2\theta}{\frac{1}{\epsilon_1} - \frac{1}{M^2}} + \frac{\sin^2\theta}{\frac{1}{\epsilon_2} - \frac{1}{M^2}} + \frac{2\cos^2\theta}{\frac{1}{\epsilon_3} - \frac{1}{M^2}} = 0 \qquad (1.111)$$

From this equation the Appleton-Hartree expression for the complex index of refraction can be obtained. (See Appendix B). The result is:

$$M^2 = 1 - \frac{X}{1 - iZ - \frac{Y_T^2}{2(1-X-iZ)} \pm \sqrt{\frac{Y_T^4}{4(1-X-iZ)^2} + Y_L^2}} \qquad (1.112)$$

where

$$X = \alpha$$
$$1 - iZ = \beta$$
$$Y = \Omega$$
$$Y_T = Y \sin\theta$$
$$Y_L = Y \cos\theta$$

and M^2 can be written as $(\mu - i\chi)^2 = R - iI$. The positive sign denotes the ordinary wave and the minus sign the extraordinary wave.

It is easy to see that

$$\mu = \frac{1}{\sqrt{2}}\sqrt{R + \sqrt{R^2+I^2}} \tag{1.113}$$

$$\eta = \frac{1}{\sqrt{2}}\sqrt{-R + \sqrt{R^2+I^2}} \tag{1.114}$$

for each sign. Hence it follows that both μ and η are always non-negative. And from the fact that the field quantities contain the factor $e^{i(\omega t - \underline{K}\cdot\underline{r})} = e^{i\omega(t - \frac{M}{c}r)}$ where it has been assumed that $\underline{n} \| \underline{r}$, then substituting for M

$$e^{i(\omega t - \frac{\omega}{c}Mr)} = e^{i[\omega t - \frac{\omega}{c}(\mu - i\eta)r]} \tag{1.115}$$

$$= e^{i(\omega t - \frac{\omega}{c}\mu r)} e^{-\frac{\omega}{c}\eta r} \tag{1.116}$$

as it is seen that apart from an oscillating factor $e^{i(\omega t - \frac{\omega}{c}\mu r)}$ all field components contain a factor $e^{-\frac{\omega}{c}\eta r}$, which, because η is positive, decreases as r increases. This term, therefore, represents an attenuation on the wave as it travels in the direction of increasing r.

$\frac{\omega}{c}\eta$ is defined as the absorption coefficient γ of the wave. The preceding analysis has been concerned with only a point in what is

presumed to be a homogeneous medium. In the ionosphere, however, both the electron density and the collision frequency are functions of height (and if one wishes to be more exact, so is the magnetic field), so that the expression given here for the fields are not strictly valid, for account must be taken of the variations mentioned. The ionosphere is visualized, in this new approach, as horizontally stratified, i.e. N and ν do not depend on x, y if z has been chosen as the height. This means that M is a function of height, so that the differentiations made at the beginning must take into account this fact. This is done by Budden (Ref: 12). The solutions are the so-called WKB solutions of the ionosphere. For the particular case of a wave horizontally polarized, i.e. $E_y \neq 0$, the solution for this component is:

$$E_y \sim A M^{-1/2} \exp\left\{-i \frac{\omega}{c} \int^z M\, dz\right\} \tag{1.117}$$

for an upgoing wave. Upon substitution of $M = \mu - i\chi$

$$E_y \sim A(\mu - i\chi)^{-1/2} \exp\left\{-i \frac{\omega}{c} \int^z \mu\, dz\right\} \exp\left\{\frac{-\omega}{c} \int^z \chi\, dz\right\} \tag{1.118}$$

and assuming that the variation of M with height is fairly small, it can be concluded that the factor which indicates the attenuation of the wave is

$$e^{-\frac{\omega}{c} \int^z \chi\, dz} \tag{1.119}$$

or, in terms of the absorption coefficient.

$$e^{-\int^z \eta dz} \tag{1.120}$$

The integral $\int^z \eta dz$ is the absorption of the wave. It is this integral which is the subject of calculation in this paper.

Now, since the Appleton-Hartree equation refers to a single point, no mention being made of how M varies from point to point, it is then assumed that it is valid at any point. At each point, then, the proper values for the parameters are substituted in the equation to obtain the proper value of M^2. In principle μ, η can be calculated then.

The condition for validity of the WKB solutions has also been given by Budden. This condition is:

$$\frac{1}{K^2}\left|\frac{3}{4}\left(\frac{1}{M^2}\frac{dM}{dz}\right)^2 - \frac{1}{2M^3}\frac{d^2M}{dz^2}\right| \ll 1 \tag{1.121}$$

It is evident that this condition fails when $M=0$, i.e. near levels of reflection. The derivative $\frac{dM}{dz}$ and $\frac{d^2M}{dz^2}$ should be fairly small for the condition to be valid; given the term $\frac{1}{K^2}$ the condition is easier to obtain at higher frequencies.

III. Generalizations of the Classical Theory

A more general approach may be taken to calculate the index of refraction of a magneto-ionic medium. The most immediate step in generalizing the theory presented in the previous Chapter is to consider more carefully the collision of the electrons with the other constituents of the medium. It was implicity assumed in the classical Appleton-Hartree theory that the collision frequency of the electrons with the neutral particles of the magneto-ionic medium was constant. Experimental results have shown, however, that for gases in general this is not true. The electron collision frequency turns out to be velocity dependent. In particular, in nitrogen, it is proportional to the square of the electron velocity.

The Sen-Wyller theory then is the generalization of the Appleton-Hartree equation to the case of non-constant collision frequency. The techniques used are similar with one major exception: in the Sen-Wyller generalization, it becomes too difficult to deal with the equations of motion directly so that use is made of Boltzmann's equation, which permits the calculation of number densities and average speeds once the electron distribution function has been found.

A. The Assumptions

In the first place the medium is assumed to be a slightly ionized Lorentz gas. This Lorentz gas has two constituents of more importance than others that might be present. These are the electrons and the neutral particles. Furthermore, the mass m, of the neutral particles

is much greater than that of the electrons m_2 . The slight ionization condition can be expressed by $n_1 >> n_2$ where n_1 , is the number density of the neutral particles and n_2 that of the electrons. It is also true that ν_{ei} , the collision frequency of the electrons against ions, is much less than ν_{ea} , the collision frequency of the electrons against neutral particles. This is what is meant by saying that the electrons and neutrals particles are the most important constituents of this medium. ν_{ea} is supposed to apply to elastic collisions.

The plasma is assumed to be uniform and under the influence of a constant magnetic field and an oscillatory electric field.

Under the conditions described above, several preliminary inferences can be made which help to solve Boltzmann's equation.

B - Preliminary Inferences

For particles with a distribution function f_2 (the electrons in the present derivation), the general Boltzmann's equation is

$$\frac{\partial f_2}{\partial t} + \underline{F} \cdot \nabla_{v_2} f_2 + \underline{v}_2 \cdot \nabla_r f_2 = Q \qquad (2.1)$$

where $\underline{F}$ is the force per unit mass on the electrons, $\underline{v}_2$ their velocity, ∇_{v_2} the gradient vector in velocity space, ∇_r the gradient in position space and Q is the collision term. Under the assumption of plasma uniformity f_2 should be independent of position. Therefore $\nabla_r f_2 = 0$, and equation (2.1) simplifies to

$$\frac{\partial f_2}{\partial t} + \underline{F} \cdot \nabla_{v_2} f_2 = Q \qquad (2.2)$$

$\underline{F}$ can be easily calculated since it represents the external forces acting on the electrons per unit mass. The magnetic force is $e_2 \underline{V}_2 \times \underline{H}_o$, and the electric force is $e_2\underline{E}\cos\omega t$, where $\underline{E}$ is time-independent. Therefore

$$\underline{F} = \frac{1}{m_2}\left(e_2\underline{V}_2 \times \underline{H}_o + e_2\underline{E}\cos\omega t\right)$$

$$= \underline{\Gamma}_2 \cos\omega t - \frac{e_2}{m_2}\,\underline{V}_2 \times \underline{H}_o \qquad (2.3)$$

where $\underline{\Gamma}_2 = \frac{e_2}{m_2}\,\underline{E}$. Hence, if equation (2.3) is substituted in equation (2.2) then the latter becomes

$$\frac{\partial f_2}{\partial t} + \left(\underline{\Gamma}_2 \cos\omega t + \frac{e_2}{m_2}\,\underline{V}_2 \times \underline{H}_o\right) \cdot \nabla_{V_2} f_2 = Q \qquad (2.4)$$

The last term which should be examined is Q, the collision term. This will be done only briefly since any calculation of Q is fairly involved. The reader, interested in these details, would do well to consult Chapman and Cowling (ref 15:343). In general, it can be said that if there are two constituents in the plasma with distribution functions f_1 and f_2, and if the collisions between particles of these constituents are elastic with relative velocity g at the time of collision then

$$Q = \iiint \left(f_1' f_2' - f_1 f_2\right) g b \, db \, d\epsilon \, d\underline{V}_1 \qquad (2.5)$$

The primes indicate the distributions after collision, b and ϵ are impact parameters analogous (though not the same) as those used in Rutherford scattering. These definitions may be found in Chapman and Cowling (Ref: 15). Finally dv_1 stands for $(dv_1)_x (dv_1)_y (dv_1)_z$, that is an element of volume in a 3-dimensional velocity space.

One final inference and the equation (2.4) should be in its most appropriate form. This inference is that f_1 should be a Maxwellian distribution since the mass of the neutral particles is much greater than that of the electron, i.e.

$$f = n_1 \left(\frac{m_1}{2\pi kT}\right)^{3/2} e^{-\frac{m_1 v_1^2}{2kT}} \qquad (2.6)$$

where n_1, is the number density of the neutral particles and the other symbols have their usual meaning. This is the reason why only one Boltzmann's equation has been written down. Normally a Boltzmann equation would have to be solved for each constituent of the plasma.

3 - The Solution of Boltzmann's Equation

The method of solution is the so called Chapman-Enskog Method. Theoretical justification for this method may be found in Chapman and Cowling (Ref 15). Here as in the Sen-Wyller paper (Ref 49) it will merely be used. The method consists of the following, f_2 is assumed to be of the form

$$f^{(1)} + [\underline{r}_2 \cdot \underline{V}_2](\alpha_2 \cos\omega t + \beta_2 \sin\omega t) + (\underline{H}_0 \times \underline{r}_2)\cdot \underline{V}_2 (\xi_2 \cos\omega t + \eta_2 \sin\omega t) + [\underline{H}_0 \times (\underline{H}_0 \times \underline{r}_2)]\cdot \underline{V}_2 (\mu_2 \cos\omega t + \delta_2 \sin\omega t)$$

(2.7)

where $f_2^{(0)}, \alpha_2, \beta_2, \xi_2, \eta_2, \gamma_2, \delta_2$ are dependent only on the speed v_2 of the electrons. The task of solving Boltzmann's equation reduces them to the determination of the functions just mentioned. If equation (2.7) is substituted in equation (2.4) and the coefficients of I_2^2, $\underline{I}_2 \cdot \underline{V}_2$, $(\underline{H}_0 \times \underline{I}_2) \cdot \underline{V}_2$ and $[\underline{H}_0 \times (\underline{H}_0 \times \underline{I}_2)] \cdot \underline{V}_2$ on both sides are equated to each other, then with suitable manipulations (see Appendix C) a system of linear simultaneous equations results. From it $\alpha_2, \beta_2, \xi_2, \eta_2, \gamma_2, \delta_2$ can be calculated. The results are:

$$\alpha_2 = \frac{-\nu}{\nu^2+\omega^2} \cdot \frac{1}{v_2} \cdot \frac{\partial f_2^{(0)}}{\partial v_2} \tag{2.8}$$

$$\beta_2 = \frac{-\omega}{\nu^2+\omega^2} \cdot \frac{1}{v_2} \cdot \frac{\partial f_2^{(0)}}{\partial v_2} \tag{2.9}$$

$$\xi_2 = \frac{e_2}{m_2} \frac{(\nu^2+s^2-\omega^2)}{[\nu^2+(\omega+s)^2][\nu^2+(\omega-s)^2]} \cdot \frac{1}{v_2} \cdot \frac{\partial f_2^{(0)}}{\partial v_2} \tag{2.10}$$

$$\eta_2 = \frac{2e_2}{m_2} \frac{\omega\nu}{[\nu^2+(\omega+s)^2][\nu^2+(\omega-s)^2]} \cdot \frac{1}{v_2} \cdot \frac{\partial f_2^{(0)}}{\partial v_2} \tag{2.11}$$

$$\gamma_2 = \frac{s^2}{H_0^2} \cdot \frac{1}{(\nu^2+\omega^2)} \cdot \frac{\nu(3\omega^2-\nu^2-s^2)}{[\nu^2+(\omega+s)^2][\nu^2+(\omega-s)^2]} \cdot \frac{1}{v_2} \cdot \frac{\partial f_2^{(0)}}{\partial v_2} \tag{2.12}$$

$$\delta_2 = \frac{-s^2}{H_0^2}\frac{1}{(\nu^2+\omega^2)}\;\frac{\omega(s^2-\omega^2+3\nu^2)}{[\nu^2+(\omega+s)^2][\nu^2+(\omega-s)^2]}\cdot\frac{1}{v_2}\cdot\frac{\partial f_2^{(0)}}{\partial v_2} \qquad (2.13)$$

The symbols have been defined in Chapter 1. Only $f_2^{(0)}$ remains to be calculated. This may be done from the following equation:

$$\frac{1}{6}\Gamma_2^2 v_2^3\left[\alpha_2 - \gamma_2' H_0^2 \sin^2\psi\right] = \frac{KT}{m_1\lambda} v_2^3 \frac{\partial f_2^{(0)}}{\partial v_2} + \frac{m_2 v_2^4}{m_1\lambda} f_2^{(0)} \qquad (2.14)$$

Where ψ is the angle between $\underline{\Gamma_2}$ and $\underline{H_0}$ and $\lambda(v_2)$ is the mean free path of an electron with speed v_2. The mean free path can be expressed in terms of collision frequency as $\frac{v_2}{\nu(v_2)}$.

If equations (2.8), (2.12) are substituted for α_2, and γ_2' in (2.14) then

$$\frac{1}{6}\Gamma_2^2 v_2^3\left\{\left[\frac{-\nu}{\nu^2+\omega^2}\cdot\frac{1}{v_2}\cdot\frac{\partial f_2^{(0)}}{\partial v_2}\right] + \frac{\nu s^2(s^2+\nu^2-3\omega^2)\sin^2\psi}{(\nu^2+\omega^2)[\nu^2+(\omega+s)^2][\nu^2+(\omega-s)^2]}\cdot\frac{1}{v_2}\cdot\frac{\partial f_2^{(0)}}{\partial v_2}\right.$$

$$= \frac{KT}{m_1\lambda} v_2^3 \frac{\partial f_2^{(0)}}{\partial v_2} + \frac{m_2 v_2^4}{m_1\lambda} f_2^{(0)} \qquad (2.15)$$

If C is defined to be

$$\frac{s^2(\nu^2+s^2-3\omega^2)\sin^2\psi}{[\nu^2+(\omega+s)^2][\nu^2+(\omega-s)^2]} \qquad (2.16)$$

and if $\lambda = \frac{v_2}{\nu}$ is used, then equation (2.15) becomes

$$-\frac{1}{6}\beta_2^2\left(\frac{1}{\nu^2+\omega^2}\right)\frac{\partial f_2^{(0)}}{\partial v_2}(1-c) = \frac{kT}{m_1}\frac{\partial f_2^{(0)}}{\partial v_2} + \frac{m_2 v_2}{m_1} f_2^{(0)} \qquad (2.17)$$

or

$$\frac{\partial f_2^{(0)}}{\partial v_2}\left[kT + \frac{m_1\beta_2^2(1-c)}{6(\nu^2+\omega^2)}\right] = -m_2 v_2 f_2^{(0)} \qquad (2.18)$$

and if $F(v_2) = \frac{m_1\beta_2^2(1-c)}{6(\nu^2+\omega^2)}$ then

$$\frac{\partial f_2^{(0)}}{\partial v_2}\left(kT + F(v_2)\right) + m_2 v_2 f_2^{(0)} = 0 \qquad (2.19)$$

This differential equation which is linear with non-constant coefficients can be easily solved, i.e

$$f_2^{(0)} = B\exp\left\{-\int_0^{v_2}\frac{m_2 v_2\, dv_2}{kT + F(v_2)}\right\} \qquad (2.20)$$

B is just the normalization constant. Several interesting facts emerge from equation (2.20). First of all, not even $f_2^{(0)}$ is Maxwellian. The deviation from a Maxwellian distribution is controlled by size of $F(V_2)$. This equation, it should be stressed,

is valid for any slightly ionized Lorentz gas. As will be seen, this will be specialized later to the ionosphere where $f_2^{(o)}$ will be assumed to be Maxwellian which implies certain relationship between the size of $F(v_2)$ and κT .

Without writing f_2 in full, it can be said that Boltzmann's equation has already been solved since the functions $\alpha_2, \beta_2, \xi_2, \eta_2, \gamma_2, \delta_2$ and $f_2^{(o)}$ of v_2 are now known.

From here on the procedure is largely similar to that in Chapter I. The next step is to calculate the conductivity and the dielectric tensor.

D – The Conductivity Tensor and The Dielectric Tensor

It may be recalled that in the previous chapter the conductivity tensor was obtained by deriving an explicit relationship for $\underline{J} = \sigma \underline{E}$. This relationship was based on the solution to the equations of motion $\underline{r} = \underline{r}(\underline{E})$. In the Sen-Wyller generalization, no such relation exists but, however, since $\underline{J} = N_2 \langle \underline{v}_2 \rangle$ where $\langle \underline{v}_2 \rangle$ is the average velocity of the electrons, and since N and $\langle \underline{v}_2 \rangle$ can be calculated from the known distribution function f_2 , then $\underline{J}$ can be calculated and presumably the calculated $\underline{J}$ can be set equal to $\kappa \underline{E}$ and σ calculated.

For convenience f_2 is written as:

$$f_2^{(o)} + \underline{f}_2^{(1)} \cdot \underline{v}_2 \qquad (2.21)$$

so that one

$$\underline{f}_2^{(1)} = \underline{\Gamma}_2 (\alpha_2 \cos\omega t + \beta_2 \sin\omega t) + (\underline{H}_0 \times \underline{\Gamma}_2)(\xi_2 \cos\omega t + \eta_2 \sin\omega t)$$

$$+ [\underline{H}_0 \times (\underline{H}_0 \times \underline{\Gamma}_2)](\gamma_2 \cos\omega t + \delta_2 \sin\omega t) \qquad (2.22)$$

The number density of the electrons is

$$N_2 = \int_V f_2 \, dV \qquad (2.23)$$

where dV is an element of volume in velocity space. Since f_2 depends on the speeds v_2 it is used to express dV in a spherical coordinate system in velocity space. Hence $dV = v_2^2 \sin\theta \, d\theta \, d\phi \, dv_2$. Again for convenience, the spherical system here chosen is a special in that it is chosen such that $\underline{f}_2^{(1)}$ is parallel to the $\theta = 0$ direction (see Fig 1.b). Therefore:

$$N_2 = \int (f_2^{(0)} + \underline{f}_2^{(1)} \cdot \underline{V}_2) v_2^2 \sin\theta \, d\theta \, d\phi \, dv_2 \qquad (2.24)$$

or

$$N_2 = 4\pi \int_0^\infty f_2^{(0)} v_2^2 \, dv_2 + \int_0^{2\pi} d\phi \int_0^\pi f_2^{(1)} \sin\theta \cos\theta \, d\theta \int_0^\infty v_2^3 \, dv_2 \qquad (2.25)$$

so that, finally,

$$N_2 = 4\pi \int_0^\infty f_2^{(0)} v_2^2 \, dv_2 \tag{2.26}$$

Interestingly enough then N_2 is determined strictly by $f_2^{(0)}$. This equation also gives the normalization condition for the constant B. Using the unnormalized distribution function then,

$$N_2 \langle \underline{v}_2 \rangle_x = \int v_{2x} f_2 \, dV \tag{2.27}$$

but, in the spherical system,

$$v_{2x} = v_2 \sin\theta \cos\phi \tag{2.28}$$

so that

$$N_2 \langle \underline{v}_2 \rangle_x = \int (v_2 \sin\theta \cos\phi)(f_2^{(0)} + f_2^{(1)} v_2 \cos\theta) v_2^2 \sin\theta \, d\theta \, d\phi \, dv_2 \tag{2.29}$$

Both integrals are 0, since $\int_0^{2\pi} \cos\phi \, d\phi = 0$. The situation is similar for the y-component. For the z-component, however,

$$N_2 \langle \underline{v}_2 \rangle_z = \int v_2 \cos\theta \, (f_2^{(0)} + f_2^{(1)} v_2 \cos\theta) v_2^2 \sin\theta \, d\theta \, d\phi \, dv_2 \tag{2.30}$$

$$= \int f_2^{(0)} v_2^3 \sin\theta \cos\theta \, d\theta \, d\phi \, dv_2 + \int f_2^{(1)} v_2^4 \sin\theta \cos^2\theta \, d\theta \, d\phi \, dv_2 \tag{2.31}$$

The first integral is 0 since $\int_0^{\pi} \sin\theta \cos\theta \, d\theta = 0$ but the second one is different; in fact, it is equal to $\frac{4\pi}{3}\int f_2^{(1)} v_2^4 \, dv_2$.

It is clear then that $\underline{J}$ is parallel to $\underline{f}_2^{(1)}$, so that if the spherical system used were a more general one in which $\underline{f}_2^{(1)}$ had an arbitrary direction then

$$N_2 \langle \underline{V}_2 \rangle = \frac{4\pi}{3}\int \underline{f}_2^{(1)} v_2^4 \, dv_2 \qquad (2.32)$$

and, therefore

$$\underline{J} = \frac{4\pi e_2}{3}\int_0^{\infty} \underline{f}_2^{(1)} v_2^4 \, dv_2 \qquad (2.33)$$

so that $\underline{f}_2^{(1)}$ determines $\underline{J}$. The partial objective stated at the beginning of this section has been accomplished; that is to calculate $\underline{J}$. Now, the final step is then to express $\frac{4\pi e_2}{3}\int_0^{\infty} \underline{f}_2^{(1)} v_2^4 \, dv_2$ as $6\underline{E}$.

If equation (2.22) is used, equation (2.33) becomes:

$$\underline{J} = \frac{4\pi e_2}{3}\left\{ \underline{\Gamma}_2 \int_0^{\infty} (\alpha_2 \cos\omega t + \beta_2 \sin\omega t) v_2^4 \, dv_2 + (\underline{H}_0 \times \underline{\Gamma}_2) \int_0^{\infty} (\xi_2 \cos\omega t + \eta_2 \sin\omega t) v_2^4 \, dv_2 \right.$$

$$\left. + [\underline{H}_0 \times (\underline{H}_0 \times \underline{\Gamma}_2)] \int_0^{\infty} (\gamma_2 \cos\omega t + \delta_2 \sin\omega t) v_2^4 \, dv_2 \right. \qquad (2.34)$$

or

$$\underline{J} = \frac{4\pi}{3} e_2 \left\{ J_1 \underline{\Gamma}_2 + J_2 (\underline{H}_0 \times \underline{\Gamma}_2) + J_3 [\underline{H}_0 \times (\underline{H}_0 \times \underline{\Gamma}_2)] \right\} \qquad (2.35)$$

where

$$J_1 = \int_0^\infty (\alpha_2 \cos\omega t + \beta_2 \sin\omega t) v_2^4 dv_2 \qquad (2.36)$$

$$J_2 = \int_0^\infty (\xi_2 \cos\omega t + \eta_2 \sin\omega t) v_2^4 dv_2 \qquad (2.37)$$

$$J_3 = \int_0^\infty (\gamma_2 \cos\omega t + \delta_2 \sin\omega t) v_2^4 dv_2 \qquad (2.38)$$

A coordinate system must be chosen to calculate σ . The coordinate system chosen here is similar to the one in Figure 1 with the exception that the propagation vector $\underline{P}$ is in the x-z plane (Fig 1.c)

If use is made of the notation developed in Chapter I, then

$$(\underline{H}_0 \times \underline{\Gamma}_2)_{ij} = \epsilon_{ijk} H_j (\underline{\Gamma}_2)_k \qquad (2.39)$$

From Figure 1.c $H_x = H_y = 0$, $H_z = H_0$, so that

$$(\underline{H}_0 \times \underline{\Gamma}_2)_i = \epsilon_{i3k} H_3 (\underline{\Gamma}_2)_k \qquad (2.40)$$

as was done in Chapter I, equation (2.40) can be interpreted as the product of matrix ($\epsilon_{i3k} H_3$) and a column vector $(\underline{\Gamma}_2)_k = \frac{e_2}{m_2} (\underline{E})_k$. Therefore, by the definition of ϵ_{i3k} , in matrix notation equation (2.40) becomes

$$[\underline{H}_0 \times \underline{\Gamma}_2] = \frac{e_2}{m_2} \begin{pmatrix} 0 & -H_3 & 0 \\ H_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \qquad (2.41)$$

it is clear that

$$\left[\underline{H}_0 \times \left(\underline{H}_0 \times \underline{\Gamma}_2\right)\right] = \begin{pmatrix} 0 & -H_3 & 0 \\ H_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (\underline{H}_0 \times \underline{\Gamma}_2)_1 \\ (\underline{H}_0 \times \underline{\Gamma}_2)_2 \\ (\underline{H}_0 \times \underline{\Gamma}_2)_3 \end{pmatrix} \tag{2.42}$$

and by (2.41)

$$\left[\underline{H}_0 \times \left(\underline{H}_0 \times \underline{\Gamma}_2\right)\right] = \frac{e_2}{m_2} \begin{pmatrix} 0 & -H_3 & 0 \\ H_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -H_3 & 0 \\ H_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \tag{2.43}$$

$$= \frac{e_2}{m_2} \begin{pmatrix} -H_3^2 & 0 & 0 \\ 0 & -H_3^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \tag{2.44}$$

so that by equation (2.35), (2.41), (2.44) then

$$\underline{J} = \frac{4\pi e_2^2}{3 m_2} \begin{pmatrix} J_1 - H_3^2 J_3 & -J_2 H_3 & 0 \\ J_2 H_3 & J_1 - H_3^2 J_3 & 0 \\ 0 & 0 & J_1 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \tag{2.45}$$

so that, finally

$$\epsilon = \frac{4\pi e_2^2}{3 m_2} \begin{pmatrix} J_1 - H_0^2 J_3 & -J_2 H_0 & 0 \\ J_2 H_0 & J_1 - H_0^2 J_3 & 0 \\ 0 & 0 & J_1 \end{pmatrix} \tag{2.46}$$

The dielectric tensor can be calculated easily enough from the definition $\hat{\epsilon}_e = \underline{I} + \frac{\sigma}{i\omega\epsilon_0}$. To simplify the writing somewhat the following definitions are useful:

$$J_i' = \frac{J_i}{N_e}$$

and

$$\epsilon_{\underline{I}} = 1 - \frac{4\pi i}{3}\frac{\omega_0^2}{\omega} J_1' \tag{2.47}$$

$$\epsilon_{\underline{II}} = \frac{-4\pi i}{3}\frac{\omega_0^2}{\omega} H_0 J_2' \tag{2.48}$$

$$\epsilon_{\underline{III}} = \frac{-4\pi}{3} i \frac{\omega_0^2}{\omega} H_0^2 J_3' \tag{2.49}$$

Using the previous definitions, then $\hat{\epsilon}_e$ becomes

$$\begin{pmatrix} \epsilon_{\underline{I}} + \epsilon_{\underline{III}} & -\epsilon_{\underline{II}} & 0 \\ \epsilon_{\underline{II}} & \epsilon_{\underline{I}} + \epsilon_{\underline{III}} & 0 \\ 0 & 0 & \epsilon_{\underline{I}} \end{pmatrix} \tag{2.50}$$

This was the main objective stated at the beginning of this section. The comments that appeared on Chapter I on the use of $\hat{\epsilon}_e$ as it stands at the moment in the dispersion relation also apply here. In Chapter I $\hat{\epsilon}_e$ was diagonalized and then the dispersion relation was obtained using the diagonal $\bar{\epsilon}_e$. To show that that is not the only simplified

procedure, in this chapter a less general one is used but which gives the same results.

E - The Dispersion Relation

It will be recalled that the wave equation is

$$(\underline{n}\cdot\underline{E})\underline{n} - (\underline{n}\cdot\underline{n})\underline{E} = \frac{\hat{\epsilon}_c}{\mu^2}\underline{E} \qquad (2.42)$$

where $\underline{n}$ is a unit vector in the direction of propagation. If the propagation is parallel to one of the coordinate axes, then $(\underline{n}\cdot\underline{E})\underline{n}$ will certainly be simplified. This is the basis for the procedure followed by Sen-Wyller (Ref 49) and this is also the reason for choosing the coordinate system in Figure 1.c, for then if the system in Figure 1.c is rotated about the y-axis as to bring the propagation vector to a direction parallel to the x-axis, then the rotation matrix of the transformation is considerably simplified.

The rotation matrix is (see Figure 1):

$$A = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix} \qquad (1.92)$$

Where the angle $\phi = \frac{\pi}{2} - \varphi$ where φ is the angle between the magnetic field and the propagation vector. The matrix A is real and orthogonal, therefore, the dielectric tensor in the new system is $A\hat{\epsilon}_c\tilde{A}$

where $\tilde{A}$ is the transpose of A.*

Therefore

$$A\epsilon_c\tilde{A} = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix} \begin{pmatrix} \epsilon_I + \epsilon_{III} & -\epsilon_{II} & 0 \\ \epsilon_{II} & \epsilon_I + \epsilon_{III} & 0 \\ 0 & 0 & \epsilon_I \end{pmatrix} \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ +\sin\phi & 0 & \cos\phi \end{pmatrix} \quad (2.52)$$

or

$$\begin{pmatrix} \epsilon_I + \epsilon_{III}\cos^2\phi & -\epsilon_{II}\cos\phi & -\epsilon_{III}\sin\phi\cos\phi \\ \epsilon_{II}\cos\phi & \epsilon_I + \epsilon_{III} & -\epsilon_{II}\sin\phi \\ -\epsilon_{III}\sin\phi\cos\phi & \epsilon_{II}\sin\phi & \epsilon_I + \epsilon_{III}\sin^2\phi \end{pmatrix} \quad (2.53)$$

The wave equation (2.92) leads to three equations

$$-(\epsilon_I + \epsilon_{III}\cos^2\phi)E_x + \epsilon_{II}\cos\phi\, E_y + \epsilon_{III}\sin\phi\cos\phi\, E_z = 0 \quad (2.54)$$

$$-(\epsilon_{II}\cos\phi)E_x + [M^2 - (\epsilon_I + \epsilon_{III})]E_y + \epsilon_{II}\sin\phi\, E_z = 0 \quad (2.55)$$

$$\epsilon_{III}\sin\phi\cos\phi\, E_x - \epsilon_{II}\sin\phi\, E_y + [M^2 - (\epsilon_I + \epsilon_{III}\sin^2\phi)]E_z = 0 \quad (2.56)$$

where E_x, E_y, E_z are the vector components of $\underline{E}$ in the new system.

---

* This seems to be contrary to what was done in Chapter I, where it would seem that ϵ_e in the new system would be $\tilde{A}\epsilon_e A$, there is no contradiction if it is observed that the term rotation matrix applies to the vectors, i.e. $E' = AE$, where E' is the vector E in the new system so that the similarity transformation on ϵ_e is actually $(A^{-1})^{-1}\epsilon_e(A^{-1})$ or $A\epsilon_e A^{-1} = A\epsilon_e\tilde{A}$ in this case.

The system of equations (2.54), (2.55) and (2.56) has a non-trivial solution only if the determinant of the system is 0 . Therefore,

$$\begin{vmatrix} -(\epsilon_I + \epsilon_{\underline{III}} \cos^2\phi) & \epsilon_{\underline{II}} \cos\phi & \epsilon_{\underline{III}} \sin\phi\cos\phi \\ -\epsilon_{\underline{II}} \cos\phi & M^2 - (\epsilon_I + \epsilon_{\underline{III}}) & \epsilon_{\underline{II}} \sin\phi \\ \epsilon_{\underline{III}} \sin\phi\cos\phi & -\epsilon_{\underline{III}} \sin\phi & M^2 - (\epsilon_I + \epsilon_{\underline{III}} \sin^2\phi) \end{vmatrix} = 0 \qquad (2.57)$$

Equation (2.57) is the dispersion relation for the Sen-Wyller generalization. Its simplification and solution for M^2 leads to (see Appendix D)

$$M^2 = (\mu - i\chi)^2 = \frac{A + B\sin^2\varphi \pm \sqrt{B^2\sin^4\varphi - C^2\cos^2\varphi}}{D + E\sin^2\varphi} \qquad (2.58)$$

where

$$A = 2\epsilon_I (\epsilon_I + \epsilon_{\underline{III}}) \qquad (2.59)$$

$$B = \epsilon_{\underline{III}} (\epsilon_I + \epsilon_{\underline{III}}) + \epsilon_{\underline{II}}^2 \qquad (2.60)$$

$$C = 2\epsilon_I \epsilon_{\underline{II}} \qquad (2.61)$$

$$D = 2\epsilon_I \qquad (2.62)$$

$$E = 2\epsilon_{\underline{III}} \qquad (2.63)$$

Equation (2.58) is the Sen-Wyller magneto-ionic formula and will be so referred to in what follows.

F - Ionospheric Application

In the ionosphere only certain kinds of velocity dependence of the collision frequency are of importance. For example, it was indicated at the beginning of the chapter that Phelps and Pack had determined that in nitrogen

$$\nu = \nu_m \frac{m_2 v_2^2}{2KT} = \nu_m \epsilon \tag{2.64}$$

Since nitrogen is the most abundant gas in the ionosphere at least at the lower heights, it is useful to compute $\epsilon_I, \epsilon_{II}, \epsilon_{III}$ for the ionosphere. If $\Gamma_2^2 << KT$, i.e. if the field applied is much less than the thermal energy, then by equation (2.16), (2.18)

$$f_2^{(0)} = A e^{-\epsilon} \tag{2.65}$$

i.e. $f_2^{(0)}$ is Maxwellian. Therefore,

$$f_2^{(0)} = n_2 \left(\frac{m_2}{2\pi KT}\right)^{3/2} e^{-\epsilon} \tag{2.66}$$

The handling of the integrals J_1, J_2 and J_3 may be made more convenient if they are written in their complex form, keeping in

mind, however, that it is the real part the one that is of interest. Their complex forms are:

$$J_1 = e^{i\omega t}\int_0^\infty \left(\alpha_2 + \frac{\beta_2}{i}\right) v_2^4\, dv_2 \tag{2.67}$$

$$J_2 = e^{i\omega t}\int_0^\infty \left(\xi_2 + \frac{\eta_2}{i}\right) v_2^4\, dv_2 \tag{2.68}$$

$$J_3 = e^{i\omega t}\int_0^\infty \left(\delta_2 + \frac{\delta_2}{i}\right) v_2^4\, dv_2 \tag{2.69}$$

It is easy to see that their real parts are in accordance with (2.36), (2.37), and (2.38). The procedure for the calculation of ϵ_I, ϵ_{II} and ϵ_{III} will be illustrated only for ϵ_I; the rest are obtained in the same way.

If α_2 and β_2 are substituted in equation (2.67) then

$$J_1 = e^{i\omega t}\int_0^\infty \left[\frac{-\nu}{\nu^2+\omega^2}\cdot\frac{1}{v_2}\cdot\frac{\partial f_2^{(0)}}{\partial v_2} + \frac{i\omega}{\nu^2+\omega^2}\cdot\frac{1}{v_2}\cdot\frac{\partial f_2^{(0)}}{\partial v_2}\right] v_2^4\, dv_2 \tag{2.70}$$

but since $f_2^{(0)} = Ae^{-\epsilon}$, $\frac{\partial f_2^{(0)}}{\partial v_2} = -\left(\frac{m_2 v_2}{KT}\right) f_2^{(0)}$. If this substitution is made in equation (2.70) and some simplification is done, then

$$J_1' = e^{i\omega t}\int_0^\infty \left[\frac{\nu_m \epsilon^{5/2}}{\nu_m^2\epsilon^2+\omega^2} - \frac{i\omega\epsilon^{3/2}}{\nu_m^2\epsilon^2+\omega^2}\right]\frac{e^{-\epsilon}}{(\pi)^{3/2}}\, d\epsilon \tag{2.71}$$

Therefore

$$\epsilon_I = 1 - \frac{4\pi i}{3\omega}\omega_0^2 J_1' = 1 - \frac{4\pi i}{3\omega}\omega_0^2 e^{i\omega t}\left[\int_0^\infty \frac{\nu_m \epsilon^{5/2}}{\nu_m^2\epsilon^2+\omega^2} - \frac{i\omega\epsilon^{3/2}}{\nu_m^2\epsilon^2+\omega^2}\right]\frac{e^{-\epsilon}}{\pi^{3/2}}\, d\epsilon \qquad (2.72)$$

If some rearrangements are made, then

$$\epsilon_I = 1 - \frac{4\omega_0^2 e^{i\omega t}}{3\nu_m^2\sqrt{\pi}}\int_0^\infty \frac{\epsilon^{3/2}e^{-\epsilon}}{\epsilon^2+\left(\frac{\omega}{\nu_m}\right)^2} d\epsilon - \frac{4i\omega_0^2 e^{i\omega t}}{3\omega\nu_m\sqrt{\pi}}\int_0^\infty \frac{\epsilon^{5/2}e^{-\epsilon}}{\epsilon^2+\left(\frac{\omega}{\nu_m}\right)^2} d\epsilon \qquad (2.73)$$

From Burke and Hara (Ref 23) it may be seen that a certain kind of integrals called the semi-conductor integrals are defined very similarly to those in equations (2.73): In particular the ρ_p semi-conductor integrals are defined thus:

$$\rho_p(x) = \frac{1}{p!}\int_0^\infty \frac{\epsilon^p e^{-\epsilon}}{\epsilon^2+x^2} d\epsilon \qquad (2.74)$$

Therefore,

$$\epsilon_I = 1 - \frac{\omega_0^2}{\nu_m^2} e^{i\omega t}\rho_{3/2}\left(\frac{\omega}{\nu_m}\right) - \frac{i5\omega_0^2}{2\omega\nu_m} e^{i\omega t}\rho_{5/2}\left(\frac{\omega}{\nu_m}\right) \qquad (2.75)$$

or

$$\epsilon_I = 1 - \frac{\omega_0^2}{\nu_m^2}\rho_{3/2}\left(\frac{\omega}{\nu_m}\right) - \frac{i5\omega_0^2}{2\omega\nu_m}\rho_{5/2}\left(\frac{\omega}{\nu_m}\right) \qquad (2.76)$$

$$\epsilon_I = 1 - a - ib \qquad (2.77)$$

Where $a = \frac{\omega_o^2}{\nu_m^2} \ell_{3/2}\left(\frac{\omega}{\nu_m}\right)$, $b = \frac{5\omega_o^2}{2\omega\nu_m} \ell_{5/2}\left(\frac{\omega}{\nu_m}\right)$. Similarly

$$\epsilon_{\underline{n}} = \tfrac{1}{2}(f-d) + \tfrac{i}{2}(c-e) \qquad (2.78)$$

$$\epsilon_{\underline{m}} = [a - \tfrac{1}{2}(c+e)] + i[b - \tfrac{1}{2}(f+d)] \qquad (2.79)$$

Where

$$c = \frac{\omega_o^2(\omega-s)}{\omega\nu_m^2} \ell_{3/2}\left(\frac{\omega-s}{\nu_m}\right) \qquad (2.80)$$

$$d = \frac{5\omega_o^2}{2\omega\nu_m} \ell_{5/2}\left(\frac{\omega-s}{\nu_m}\right) \qquad (2.81)$$

$$e = \frac{\omega_o^2(\omega+s)}{\omega\nu_m^2} \ell_{3/2}\left(\frac{\omega+s}{\nu_m}\right) \qquad (2.82)$$

$$f = \frac{5\omega_o^2}{2\omega\nu_m} \ell_{5/2}\left(\frac{\omega+s}{\nu_m}\right) \qquad (2.83)$$

These constants a, b, c, d, e, f are calculated once the profiles of collision frequency and electron density are known and therefore determine $\epsilon_I, \epsilon_{II}, \epsilon_{\underline{m}}$, which in turn will determine A, B, C, D, and E in the Sen-Wyller Magneto-ionic formula. So then μ, γ can be determined at any point in the ionosphere.

A Note on Shkarofsky's Generalization

In this Chapter, the classical Appleton-Hartree theory was modified to include a collision frequency which was dependent on velocity. Specifically, the results were specialized to the ionosphere by using experimental results by Phelps and Pack which showed that for nitrogen, the collision frequency ν was proportional to the square of the speed the electrons. The assumption of a slightly ionized gas was retained throughout the Sen-Wyller generalization, but since the main interest that this theory has for the author is its application to the ionosphere, it becomes clear that for the F-layer and above where the number of ions has increased considerably, a slightly ionized gas is not an adequate model of the ionosphere for these heights. Hence, an approach which could consider any degree of ionization in the ionosphere would make the ideal approach. This approach was undertaken by Shkarofsky. Here the author outlines Shkarofsky's approach, shows that certain approximations may be used and justifies a final approach for the ionospheric absorption calculations.

As presented in the first two chapters any generalization of the Appleton-Hartree equation must undertake to generalize the conductivity tensor. This is therefore the first step in Shkarofsky's generalization, the generalized conductivity tensor obtained by him is (Ref 50):

$$\epsilon = \begin{pmatrix} b+c & i(b-c) & 0 \\ -i(b-c) & b+c & 0 \\ 0 & 0 & d \end{pmatrix}$$

Where $$d = \frac{Ne^2}{m} \frac{1}{\langle \nu_g \rangle g_o + i\omega h_o}$$

$$2c = \frac{Ne^2}{m} \frac{1}{\langle \nu_g \rangle g_+ + i(\omega+s) h_+}$$

$$2b = \frac{Ne^2}{m} \frac{1}{\langle \nu_g \rangle g_- + i(\omega-s) h_-}$$

where $\langle \nu_g \rangle$ is the total average collision frequency of the electron and, $g_{o,+,-}$ and $h_{o,+,-}$ are correction factors to account for the variation with velocity of the electron - neutral particle collision frequency and for electron-ion and electron-electron effects.

From the relationship

$$\epsilon_c = \frac{\sigma}{i\omega\epsilon_o} + I$$

between the dielectric tensor and the conductivity tensor it is easy to obtain the former and if it is substituted in the dispersion relation it is a matter of algebraic manipulation to obtain a new equation for the refractive index. The equation, written in full, is:

$$M^2 = 1 - \frac{X + \left[\frac{-iX(Z\Theta_1 + i\Theta_3)}{h_o - X - ig_o Z}\right] \sin^2\theta}{J_1 - iZJ_2 + \left\{\left(\frac{-YJ_3 + Z^2\Theta_5 + iZ\Theta_4 + iX(Z\Theta_1 + i\Theta_3)}{2(h_o - X - ig_o Z)}\right) \sin^2\theta \pm \left[\left(\frac{-YJ_3 + Z^2\Theta_5 + iZ\Theta_4 - iX(Z\Theta_1 + i\Theta_3)}{2(h_o - X - ig_o Z)}\right)^2 \sin^4\theta + (YJ_4 - iZ\Theta_2)^2 \cos^2\theta\right]^{1/2}\right\}}$$

where X, Y, Z are the same parameters used in the first chapter; and the θ and λ functions are defined as follows:

$$2\theta_1 = 2g_0 - g_- + g_+$$

$$2\theta_2 = g_+ - g_-$$

$$2\theta_3 = 2h_0 - h_- - h_+ - Y(h_+ - h_-)$$

$$2\theta_4 = g_0(h_- + h_+) + h_0(g_- + g_+) - 2g_- h_+ - 2g_+ h_- + Y(-g_0 h_- + g_0 h_+ - 2g_- h_+ + 2g_+ h_-)$$

$$2\theta_5 = g_0 g_- + g_0 g_+ - 2g_- g_+$$

$$2\lambda_1 = h_+ + h_- + Y(h_+ - h_-)$$

$$2\lambda_2 = g_+ + g_-$$

$$2Y^2\lambda_3 = h_0(h_- + h_+) - 2h_- h_+ + Yh_0(h_+ - h_-) + 2h_- h_+ Y^2$$

$$2Y\lambda_4 = h_+ - h_- + Y(h_+ + \ldots)$$

and g_0, h_0 are the values of g, h for argument $\omega/\langle\nu_g\rangle$; g_+, h_+ for argument $(\omega+s)/\langle\nu_g\rangle$, and g_-, h_- for argument $\langle\omega-s\rangle/\langle\nu_g\rangle$.

The behavior of these functions should be explained. For a given value of $\langle\nu_m\rangle/\langle\nu_{ei}\rangle$ (the ratio of collisions with atoms over collisions with ions, which indicates the degree of ionization of the gas; for high values the gas is slightly ionized, low values tend to indicate fully ionized gases) the arguments are made to vary; g and h are then seen to be monotonic; for small values of $(\omega \pm s)/\langle\nu_g\rangle$, g increases from a limiting value less than unity to unity at large

values of $(\omega \pm s)/\langle \nu_g \rangle$, while h decreases from a limiting value greater than unity to unity.

The behavior with respect to $\langle \nu_m \rangle / \langle \nu_{ei} \rangle$ is somewhat more ccmplicated. It is related to the velocity dependence of the electron-neutral particle collision frequency ν_m,

$$\nu_m = c v^r$$

where v is the velocity of the electrons; if $r \leq 0$, g and h have a monotonic variation with $\langle \nu_m \rangle / \langle \nu_{ei} \rangle$; if $r = 0$, g and h, are unity for large values of $\langle \nu_m \rangle / \langle \nu_{ei} \rangle$; and if $r \geq 1$, g and h vary from the values for the slightly ionized gas to approximately unity, at $\langle \nu_m \rangle / \langle \nu_{ei} \rangle$ between 1 and 10, and then g and h deviate more from unity as $\langle \nu_m \rangle / \langle \nu_{ei} \rangle$ decreases, until they approach the limits of the completely ionized gas.

Hence, if $\omega_L = s \gg \langle \nu_g \rangle$ and $\omega > s$, then all the g and h functions approach 1 and the usual Appleton-Hartree equations result with $Z = \frac{\langle \nu_g \rangle}{\omega}$ how the conditions cited here can apply to the F region of the ionosphere. Shkarofsky has determined as a good estimate that if

$$s \geq 6 \times 10^3 \langle \nu_g \rangle$$

then investigation of the θ and α curves reveal that it is safe to use the ordinary Appleton-Hartree equation provided that

$$\frac{s}{\omega} \leq 0.9 \quad , \quad 100 \leq \frac{s}{\omega} \leq 2$$

In the present paper all the frequencies used are higher than 2.5 Mc/s so that the first condition is always satisfied. Hence, we can combine the first two approaches to describe a variable state of ionization in the ionosphere: below the heights at which the aforementioned conditions begin to apply the Sen-Wyller theory should provide a valid description of the ionosphere and at heights where the condition applies, the ordinary Appleton-Hartree equation might be used.

IV. NUMERICAL APPLICATIONS

In this Chapter the procedure followed will be described. For the sake of illustration, the example of analytical profiles for the electron density $N(h)$ and collision frequency $\nu(h)$ will be used to compute absorption and the frequency dependence of it. The particular profile of $N(h)$ used as an example is the Chapman-layer profile although there exists some other profiles which lend themselves to easy computations. The Chapman-layer is however, the more realistic and useful. For a discussion of the other profiles, Budden (Ref 12: Ch 10) may be consulted.

After the examination of this particular profile, the profiles used in the present calculations will be introduced and some words as to the manner of their selection will be set down. A discussion of several difficulties which are encountered in any calculation of this type will be given. Also, possible errors and systematic sources of inaccuracies will be discussed along with certain effects which are neglected in the computations.

A - <u>The General Line</u>

As mentioned at the end of the first chapter, the fundamental equation in these calculations is

$$A(DB) = 4.33 \int \kappa \, ds \qquad (3.1)$$

Where A is the absorption of the wave in db, ds the path of the wave and κ the absorption coefficient. For each sign in the Appleton-

Hartree or the Sen-Wyller equations a different η can be determined. These will be designated by η_o and η_x, the first for the ordinary ray and the second for the extrodinary. The differential absorption is defined as:

$$D = A_x - A_o \tag{3.2}$$

Therefore

$$D = 4.33 \int (\eta_x - \eta_o)\, ds \tag{3.3}$$

One of the objectives of this paper is to determine the frequency dependence of A_x, A_o and D. Two things are necessary for the calculation of these integrals: know the path of the ray and $\eta_{x,o}$ as a function of the path. Since the usual experiments restrict themselves to the vertical direction, the path and the height may be said to be equivalent. This point needs a little more discussion, for it is not quite so simple as that, but it will be posponed until later after the general direction of the calculations has been outlined. The other, the determination of $\eta_{x,o}$ as an analytic function of height, is something that may be done only in the simplest of cases. This will be illustrated in the example included in this chapter. It is convenient to express η as a function of the real and imaginary parts of the complex index of refraction which was the quantity derived in

in the first two chapters. This has already being done. The equations are:

$$\mu = \frac{1}{\sqrt{2}} \sqrt{R + \sqrt{R^2 + I^2}} \qquad (1.113)$$

$$\gamma = \frac{1}{\sqrt{2}} \sqrt{-R + \sqrt{R^2 + I^2}} \qquad (1.114)$$

where R is the real part of M^2 and I the imaginary. So that

$$A = 4.33 \int K(R, I)\, dL \qquad (3.4)$$

K and I are, of course, functions of height.

Another definition is of usefulness at this point. In many instances A may be written as

$$4.33 \int K N(L)\, dL \qquad (3.5)$$

Where K = K(v,w) independent since it represents the absorption per electron per kilometer of path as a function of height. When this separation is possible considerable labor is saved since K can be calculated independently of the electron density profile.

Lastly, the limits of the integrals are taken according to the region of interest. For the total traversal of the ionosphere the

the upper limit is taken according to the estimates of the remaining ionization higher up. A cutoff point is selected on the basis of the estimated error which would result if the upper limit were chosen at a certain particular height.

B - Chapman-Layer Absorption for the QL Approximation in the A-H Theory

In this example (Ref 27) some features of the absorption calculations are brought out. It serves as a guide for the more complicated calculations which follow.

For the QL approximation in the Appleton-Hartree equation:

$$M^2 = 1 - \frac{X}{1 - iZ \pm Y_L} \qquad (3.6)$$

Where the + sign denotes the ordinary ray, the - sign the extraordinary ray. Equation (3.6) may be put in the form

$$M^2 = \mu^2 - \eta^2 - 2\mu\eta i$$

$$= 1 - \frac{X(1 \pm Y_L)}{(1 \pm Y_L)^2 + Z^2} - i\frac{ZX}{(1 \pm Y_L)^2 + Z^2} \qquad (3.7)$$

If $\omega >> \omega_c$ for the particular layer being examined, then $\mu \approx 1$ (the absorption undergone when $\mu \approx 1$ is called non-deviative absorption) and $M \approx 2\chi$. Therefore,

$$\eta = \frac{ZX}{(1 \pm Y_L)^2 + Z^2} \qquad (3.8)$$

In terms of frequencies the absorption coefficient is:

$$\gamma = \frac{\nu \omega_0^2}{c\left[(\omega \pm S)^2 + \nu^2\right]} \qquad (3.9)$$

If it is recalled that $\omega_0^2 = \frac{Ne^2}{\epsilon_0 m}$ and if $\omega \pm S = \omega_e$ (the effective operating frequency), then

$$\gamma = \frac{e^2}{\epsilon_0 c m} \frac{N\nu}{\left[\omega_e^2 + \nu^2\right]} \qquad (3.10)$$

The specific absorption is

$$K = \frac{e^2}{\epsilon_0 c m} \frac{\nu}{\left[\omega_e^2 + \nu^2\right]} \qquad (3.11)$$

which is a function of height because ν and S functions of height. It should be stressed that in this approximation the collision frequency is not negligible when compared to the effective operating frequency. Because of this reason this equation could be applied to D-region absorption when the other conditions of high frequency (compared to the critical frequency) obtain. It may also be seen in what way profiles of electron density and collision frequency are needed for the computations of absorption. For a realistic atmosphere these profiles have no analytical expression, as mentioned

in the introduction.

Some useful computations may be performed if a particular ionized layer is assumed to be of the Chapman-Layer type, i.e.

$$N(h) = N_o \exp \tfrac{1}{2}(1 - z - \sec\chi\, e^{-z}) \tag{3.12}$$

Where N_o is the peak electron density when the sun is at the zenith, χ is the zenith angle and $z = \frac{h-h_o}{H}$. The collision frequency is assumed to have the profile $\nu = \nu_o e^{-z}$ where ν_o is the collision frequency at $h = h_o$.

If $N(h)$ and $\nu(h)$ are substituted in equation (3.10) then

$$\kappa = \frac{e^2}{\epsilon_o c m} \frac{N_o \exp \tfrac{1}{2}(1 - z - \sec\chi\, e^{-z})\nu_o e^{-z}}{[\omega_e^2 + \nu_o^2 e^{-2z}]}$$

$$\kappa = \frac{e^2}{\epsilon_o c m} \frac{N_o \nu_o \exp(\tfrac{1}{2} - \tfrac{3}{2}z - \sec\chi\, e^{-z})}{[\omega_e^2 + \nu_o^2 e^{-2z}]} \tag{3.13}$$

Since χ is now explicitly a function of z, there should be a change of variable from h to z in equation (3.4). Since $z = \frac{h-h_o}{H}$ then $dh = H dz$. Therefore

$$A = 4.33 \int \kappa\, dh = 4.33 \frac{e^2 N_o \nu_o H}{\epsilon_o c m} \int \frac{\exp(\tfrac{1}{2} - \tfrac{3}{2}z - \tfrac{1}{2}\sec\chi\, e^{-z})}{[\omega_e^2 + \nu_o^2 e^{-2z}]}\, dz \tag{3.14}$$

The operating frequency ω_e is a constant if the gyrofrequency is constant. In the following steps it will be assumed constant.

If the substitution $y = (\frac{1}{2}\sec\gamma)^{1/2} e^{-z/2}$ is made the integral can be simplified considerably. A is then

$$4.33\left(\frac{\epsilon^2 N_0 \nu_0 H}{\epsilon_0 c m}\right)\int \frac{-\sqrt{2}\exp(i)\cos^{3/2}\gamma\, y^2 e^{-y^2}\beta^2}{(y^2+\beta^2)\omega_e^2}\, dy$$

where

$$\beta = \frac{\omega_e \sec\gamma}{2\nu_0}$$

or

$$-4.33\,\frac{\omega_e^2 \nu_0 H}{c\omega_e^2}\int \cos^{3/2}\gamma\left(\sqrt{2}\exp(i)\,\beta^2\right)\frac{y^2 e^{-y^2}}{y^4+\beta^2}\, dy \qquad (3.15)$$

Since z goes from $-\infty$ to ∞ . y goes from ∞ to 0 .
Therefore equation (3.15) can be written as

$$A = 4.33\,\frac{\omega_e^2 \nu_0 H}{c\omega_e^2}(\sec\gamma)^{-3/2} I \qquad (3.16)$$

Where

$$I = \beta^2\sqrt{2}\exp(i)\int_0^\infty \frac{y^2 e^{-y^2} dy}{y^4+\beta^2}$$

If $\epsilon = y^2$

$$I = \frac{\beta^2\sqrt{2}\exp(i)}{2}\int_0^\infty \frac{\epsilon^{1/2} e^{-\epsilon}}{\epsilon^2+\beta^2}\, d\epsilon \qquad (3.17)$$

$$I = \beta^2 \sqrt{\frac{\pi e}{2}}\, \ell_{1/2}(\beta) \qquad (3.18)$$

Where $\ell_\rho(x)$ is defined as in the second chapter. It is one of the semi-conductor integrals. These have been tabulated by Burke and Hara (Ref 23). For some of the ℓ_ρ integrals (the ones with p = 1.5 and 2.5) polynominal approximations have been derived empirically by means of computers. The author, however, has not seen one for p = .5. The usual thing in this case is to express $\ell_{1/2}(\beta)$ as a function of the Fresnel integrals (for details see Jaeger (Ref 27) and to evaluate these numerically. The advantage of the polynomial expressions, apart from their ease in computation, is that they allow the frequency dependence of absorption to be examined more clearly. At first sight

$$A \propto \frac{1}{\omega_e^2} \qquad (3.19)$$

Where ω_e is the effective operating frequency, but in reality, of course,

$$A \propto \frac{1}{\omega_e^2}\, I(\omega_e) \qquad (3.20)$$

According to Jaeger's tabulation of $I(\beta)$

$$I(\beta) = I\left(\frac{\omega_e \sec\gamma}{2\nu_e}\right) \sim \text{constant} \qquad (3.21)$$

when $\beta > 20$. Therefore, when $\omega_e > \frac{40\nu_0}{\sec\chi}$, then

$$A \propto \frac{1}{\omega_e^2} \tag{3.19}$$

but before that condition is met equation (3.20) is the one that is valid. Interestingly enough, in terms of $C_p(\beta)$, equation (3.20) is

$$A \propto \frac{1}{\omega_e^2}\left(\omega_e^2 \ell_{1/2}(\omega_e)\right) \tag{3.22}$$

$$A \propto \ell_{1/2}(\omega_e) \tag{3.23}$$

The differential absorption would be proportional to the difference of the $\ell_{1/2}$ integrals for the corresponding ω_e .

For other profiles, it is convenient to calculate $K(h)$ and then multiply it by the particular $N(h)$ at the height involved. From (3.11)

$$K = \frac{e^2}{\epsilon_0 cm}\,\frac{\nu_0 e^{-z}}{\left[\omega_e^2 + \nu_0^2 e^{-2z}\right]} \tag{3.24}$$

Differentiation shows that this function K has a maximum when

$$\omega_e = \nu \tag{3.25}$$

Since $\omega_2 = \omega \mp S$, equation (3.25) reveals that at the heights at which K is maximum for the two magnetoionic components,

$$\omega_o = \nu_1 - S \tag{3.26}$$

$$\omega_x = \nu_2 + S \tag{3.27}$$

which means that $\nu_1 > \nu_2$. This implies that the maximum of K is higher for the extraordinary ray then for the ordinary since it requires a lower collition frequency than the ordinary ray to satisfy condition (3.26)

Another very interesting fact can be obtained from equation (3.10). If the absorption coefficients for the extraordinary and ordinary rays are denoted by κ_x and κ_o respectively by denoting the respective operating frequencies by ω_{ex} and ω_{e_o}, then

$$\frac{\kappa_x}{\kappa_o} = \frac{[\omega_{e_o}^2 + \nu^2]}{[\omega_{ex}^2 + \nu^2]} \tag{3.28}$$

which is independent of N. When κ_x and κ_o are plotted on logarithmic scale

$$\log \kappa_x - \log \kappa_o = \log[\omega_{e_o}^2 + \nu^2] - \log[\omega_{ex}^2 + \nu^2] \tag{3.29}$$

and, furthermore, when $\nu \ll \omega_e$ then $\log \kappa_x - \log \kappa_o$ is nearly constant.

It should be stressed that the previous equations are strictly valid only under the specific conditions under which they were derived. In the ionosphere they will not always be valid but there are certain ranges of heights in which they can be used to simplify the analysis.

C - Some Special Cases in the Sen-Wyller Theory

For longitudinal propagation, the Sen-Wyller magneto-ionic formula reduces to

$$M^2 = \left\{1 - \frac{\omega_0^2(\omega \pm s)}{\omega \nu_m^2} P_{3/2}\left(\frac{\omega \pm s}{\nu_m}\right)\right\} - i\left\{\frac{s\omega_0^2}{2\omega\nu_m} P_{s/2}\left(\frac{\omega \pm s}{\nu_m}\right)\right\} \qquad (3.30)$$

If the frequencies ω are high, then $\mu \sim 1$, and

$$2\chi = \frac{s\omega_0^2}{2\omega\nu_m} P_{s/2}\left(\frac{\omega \pm s}{\nu_m}\right) \qquad (3.31)$$

$$\gamma = \frac{s\omega_0^2}{4c\nu_m} P_{s/2}\left(\frac{\omega \pm s}{\nu_m}\right) \qquad (3.32)$$

The specific absorption function $\kappa = \frac{se^2}{4c\epsilon_0 m\nu_m} P_{s/2}\left(\frac{\omega \pm s}{\nu_m}\right)$, which is again independent of N. Another fact which should be stressed is that

$$\frac{\gamma_x}{\gamma_0} = \frac{P_{s/2}\left(\frac{\omega - s}{\nu_m}\right)}{P_{s/2}\left(\frac{\omega + s}{\nu_m}\right)} \qquad (3.33)$$

or $\frac{C_{s/2}\left(\frac{\omega_{ex}}{\nu_m}\right)}{C_{s/2}\left(\frac{\omega_{eo}}{\nu_m}\right)}$ in terms of the effective operating frequencies. So that judging from Burke and Hara (Ref 23) for high altitudes where is very low, $\frac{\gamma_x}{\gamma_o}$ should be nearly a constant. (See Fig 2). There are other valuable approximations which can be made but they are not useful in this case. For a list of this approximations Ratcliffe (Ref 45) can be consulted along with the original papers.

D - Selection of Profiles

Perhaps the previous sections have given an indication of two ways that one can proceed to compute absorption in the ionosphere. One can either select several profiles for electron density and a collision frequency profile or one can compute the specific absorption function and then multiply by the particular profile. The second procedure has the advantage that the once the $\mathcal{K}$ function has been obtained one can multiply it by any desired profile and obtain a quick idea of the variation of the absorption coefficient but then, for complicated ionospheric conditions there is no guarantee that K is independent of . The previous sections indicate that K is certainly independent of N for longitudinal propagation but one has to be careful that this condition obtains to use it for any particular interpretation.

Both procedures were followed in this paper. First the selection of the profiles used will be described, and in a later section some mention will be made of the other procedure.

The profiles of electron density and collision frequency selected

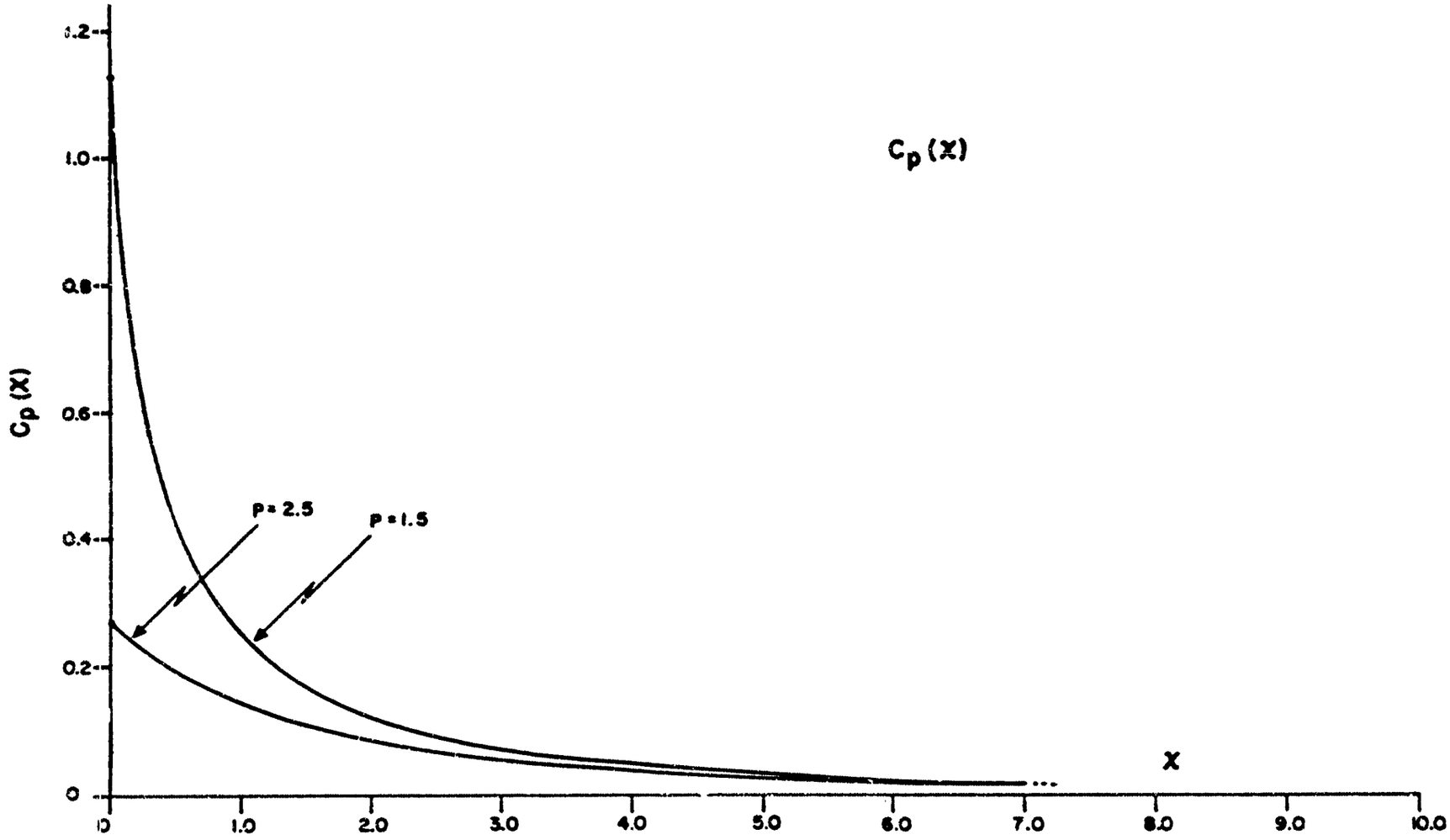


Fig. 2 (Ref 23)

for computation are necessarily of a representative nature. Among the reasons for this are the inaccuracies present in the actual measurements of these parameters in the ionosphere, the complex set of conditions that determines these parameters, and daily, monthly, and other variations which result in a large spread of values for N and γ. Furthermore, in addition to being merely representative they are also necessarily extrapolated in certain cases; the author, for example, knows of no single experiment which has measured N throughout the whole ionosphere. Therefore several groups of data from different experimenters were tied together by appropriate extrapolation to form a composite profile which is assumed representative throughout the ionosphere.

For the daytime profile, the lower heights were covered by a profile derived by Thrane and Piggott* at Kjeller, Norway on the basis of cross-modulation experiments. The profile expecially applies to noon-equinox conditions at sunspot maximum. Values from this profile were plotted up to a height of 100 km. From heights between 100 and 200 a profile by A. K. Paul and J. W. Wright (Ref 43) obtained at Eglin Air Force Base, Florida, was used. These authors have calculated several profiles for different hours of the day. The one selected was a noon profile which most smoothly fitted the one at Kjeller; the range of electron density in the heights where there was overlapping was smoothed out by a very small change in $\frac{dN}{dh}$ at the end point in the

* Unpublished paper.

curve of Thrane. The rest of the daytime profile is obtained from a profile by Shkarofsky (Ref 50) up to a height of 340 Km. Beyond 340 Km an exponential decrease of N is assumed. This assumption is corroborated by topside sounder data for dip angles of approximately 80° (See LeGalley) (Ref 33:192).

The results of two authors were used for collision frequency during the day: those of Nicholet (Ref 39) and Shkarofsky (Ref 50). Those of Nicolet were used for the lower heights (up to about 100 Km) and for the rest of the heights those values given by Shkarofsky. A word of caution should be added: the curves given for collision frequency of the electrons include collisions with ions which became of importance in the F-region; the Appleton-Hartree and the Sen-Wyller formula, however, include only electron-neutral collisions; since in the lower ionosphere the electron-ion collision frequency is negligible compared to the electron-neutral collision frequency, the Appleton-Hartree and the Sen-Wyller equations are valid (with reference to collision frequency considerations) for the lower ionosphere. As the height increases, they become less valid. Obviously, the variations of ν_{ei} and ν_{ea} with height are partly a result of the varying degree of ionization of the ionosphere as the height varies. A derivation of the magneto-ionic formulas which would recognize this fact would be preferable. This approach has been tried by Shkarofsky (Ref 50) (See note included in this paper).

The night-time $N(h)$ profile was selected from an NBS Technical Note (Ref 56). The profile was derived for the month of January 1960

at Puerto Rico. The values of $N(h)$ are provided in the Note in tabular form for the 24 hours of the day. Of these, the 0400 o'clock profile was selected as it was that one with the smallest peak in electron density and so deviated the most from a daytime profile. The tabular values were plotted and joined by straight line segments. With this approximation in mind, a new table of values was made for $N(h)$ which filled the gaps the other had.

The night-time collision frequency was more difficult to get. Night-time collision frequency profiles are, in fact, not readily available in the literature. A profile had to be assumed which would provide a reasonable basis for night-time calculations of the index of refraction. For heights below 200 km the same exponential decrease was used. Above 200 Km the night-time $N(h)$ profile was used to recalculate collision frequency values according to formulas given by Shkarofsky (Ref 50). The two curves were joined by extrapolation following, to a certain extent, the shape of the day time profile.

As can be seen, the selection of these models was to a certain extent arbitrary. They serve, however, a purpose, i.e. to bring out of the formulas certain regularities or features which are independent of the model chosen. The conclusions are then relative things, comparisions, frequency dependence relations, etc.

E - The Calculations

The calculations are based on the following two formulas derived in the first and second chapters, the Appleton-Hartree formula:

$$M^2 = 1 - \frac{X}{1 - iZ - \frac{Y_T^2}{2(1-X-iZ)} \pm \sqrt{\frac{Y_T^4}{4(1-X-iZ)^2} + Y_L^2}}$$

and the Sen-Wyller magneto-ionic formula:

$$M^2 = \frac{A + B\sin^2\varphi \pm \sqrt{B^2\sin^4\varphi - C^2\cos^2\varphi}}{D + E\sin^2\varphi}$$

It was previously stated that if M^2 was written as $R - iI$ where R and $-I$ are the real and imaginary parts of M^2 then

$$\mu = \frac{1}{\sqrt{2}}\sqrt{R + \sqrt{R^2 + I^2}}$$

$$\eta = \frac{1}{\sqrt{2}}\sqrt{-R + \sqrt{R^2 + I^2}}$$

R and I are in general complicated functions of N and ν through which they depend on the height, except, of course, in cases as simple as the one presented in the beginning sections of this chapter. Numerical calculations had, therefore, to be undertaken. Two different programs were used (one for the Appleton-Hartree and the other for the Sen-Wyller formula) to separate M^2 into its real and imaginary part. Given a certain geographical location the angle of propagation between a vertically incident wave and the magnetic field can be determined, which, if there is no deviation, is constant for a dipole field (See Appendix D); $Y = \frac{S}{\omega}$ can be determined at any height if the variation of gyrofrequency for a dipole field is included, i.e.

$$S = S_0 \left(\frac{r_e}{r_e + h}\right)^3$$

Where f_0 is the electron gyrofrequency at ground level, r_0 the radius of the earth and for the height h, X and Z can be determined, from the collision frequency and electron density profiles respectively, as functions of height. For each height then, R and I can be computed.

For the Sen-Wyller formula the same is true, although previous tabulation of the semi-conductor integrals $C_p^n(x) = \frac{1}{p!}\int_0^\infty \frac{\epsilon^p e^{-\epsilon}}{\epsilon^2+x^2}\,d\epsilon$ is necessary. These can be tabulated from formulas given by Burke and Hara (ref 23:5-6).

With the purpose of allowing the frequency dependence of certain quantities to be brought out more explicitly, a wide range of frequencies was used for computation. These frequencies were 2, 5, 3, 5, 7, 10, 14, 20, 40, 80 $^{mc}/_{s}$. Some of these were examined under both day-time and night-time conditions.

Once μ, η are computed as functions of height then numerical integrations may be performed on them to calculate quantities defined at the beginning of this chapter.

Finally, the geographical location used (Sagamore Hill, Mass) is consistent with $\theta = 73°$ and $f_0 = 1.57$ $^{mc}/_{s}$, and Y_T, Y_L are computed from these for any frequency and height.

F - Note on Possible Sources of Inaccuracies

The calculations done by the simple method explained above, involve no approximations so far as $\eta(h)$ and $\mu(h)$ are concerned. But it will be recalled that to evaluate the integrals the approximation was made that the path of the wave could be set equal to the height, this

approximation is used throughout the literature and is valid for high-frequency waves. Without going into much detail, however, it may be mentioned that there occurs always some deviation in the path of the ray. This deviation becomes particularly important when $\mu\nu'$ becomes much less than unity (i.e. for low frequencies). Since the interest of the present paper is on waves which penetrate the ionosphere (see Int.), it should be stressed that the curves computed for 2.5 mc/s , 5 mc/s , 7 mc/s can not be integrated throughout the whole ionosphere since these are reflected somewhere in the ionosphere. It was also assumed in these calculations that the field of the earth was a dipole field. This is only approximately true, a first order approximation as extensive investigations (Ref 33: Ch. 9) of the geomagnetic field show. A third factor which hinders the accuracy of the computations is the accuracy of the electron density and collision frequency profiles, which are determined by several different methods under varying conditions. The best one can hope to do is to use profiles for average representative conditions such as mean noon electron density profile, etc.

A fourth factor which is implicit in the above calculations and which also effects the accuracy is the Sen-Wyller approach itself. As was already mentioned, the Sen-Wyller approach is strictly valid for a slightly ionized Lorentz gas. The ionosphere will satisfy this condition at the lower heights but probably not at the F-region. Shkarofsky's generalization considers all degrees of ionization. Unfortunately certain

functions (ϕ and $\sqrt{}$) in his formulation are, admittedly, not accurate enough for a height profile computation of the index of refraction (i.e. for μ and χ). They do, however, indicate certain regions where all formulations are still valid (see end of Ch. 2).

V. RESULTING HEIGHT PROFILES OF μ , γ AND ASSOCIATED RESULTS

The results of the different computations performed are presented in this Chapter. They represent, in the first place, height profiles for the integrands for ordinary and extraordinary absorption. Also some profiles for μ_x have been presented. The computed profiles were graphed automatically, and are presented in Appendix F. A stands for $N(L)$, B for $\nu(L)$, U_4 for μ_o , and χ_-, $\chi_{x,o}$; the log to the base 10 of each of them has been plotted against height. The main purpose for including them is because they are necessary for the calculation of the absorption integrals, but the height profiles of μ, and γ are of interest in themselves. The interpretation of the shape of these curves is not a simple matter, and it, being not strictly germane to the present calculations, has not been done extensively, but in the main has been restricted to several general statements.

In the second place, the curves have been also grouped into significant patterns to permit easier discussion. These are presented in Appendix G.

Finally, some curves of absorption coefficients, exploring its proportionality to $N(L)$, if it exists, have been presented in Appendix H.

A - General Description

The values of μ and $\gamma(\frac{e}{\omega}\gamma)$ computed by using the Sen-Wyller program were plotted as functions of height for both day and night-time conditions. Not all of the possible $\mu_x, \mu_o, \gamma_x, \gamma_o$ have been plotted. The frequencies used were 2.5, 5, 7, 10, 14 and 20 $^{m}/_{s}$. See Appendix F.

Of the curves plotted, several broad features may be noted. For night-time conditions, and for frequencies above the critical frequency of the F-layer, (5.6 mc/s) X+ and X- are seen to follow, in general, the shape of the electron density profile. It is also seen that it is the extraordinary ray which first departs from this simple behaviour as the graphs for X+, X- at 5 mc/s show: while X+ still has a variation similar to the one pointed out above while X- has a fairly "thick" and distinctive peak. For frequencies below the critical frequency they show marked behaviour with height, the behaviour of X- being the more complicated.

Correspondingly, the curves for μ+ show a distinct behaviour at the frequency changes. For frequencies above the critical frequency they are very nearly constant (approximately 1) but as the frequency of the wave decreases, a slight depression begins to appear in the curves of μ+ near 300 km until, for frequencies as low as 2.5 mc/s this depression has turned itself into a valley of low μ+ values. The curves for μ serve to indicate regions of deviative absorption.

Under day-time conditions the behaviour of the plotted curves is much more complicated, but again broad features may be noted. For frequencies above the critical frequency of the ionosphere as determined from the day-time profile, (about 9.5 mc/s), the behaviour of X+ and X- (with the exception of X- for 10 mc/s) above 300 km is similar to that of the electron density profile. Below that height they show two smaller peaks at low heights. For frequencies below the critical frequency, the behaviour of X+, X- is much too complicated to be

described by words, but again η_x is the more complicated curve and is characterized by extremely sharp peaks at certain heights.

The μ_o curves show, in day-time conditions: the μ's for the high frequencies are fairly constant, while for decreasing frequencies they tend to dip at the region of maximum electron density.

A limited comparison of these features may be found in Appendix G, where η_x, η_o, μ_x and μ_o have been plotted together for frequencies near the critical frequency and above for both sets of data. The reasons for using only these is that the integrations for absorption were going to be performed only for frequencies which are able to penetrate the whole ionosphere.

B - Absorption Per Electron

Another calculation for which results have been obtained is the investigation of the relation between the absorption per electron and the electron density. In a previous chapter, it was argued that if $\eta = KN$, where K was a function only of frequency and collision frequency, then the calculation of absorption became all the easier. For longitudinal propagation, this was shown to be true but for general directions this may not be true. Therefore, a start was undertaken towards investigating this problem.

In particular $\eta(\nu, N) = \eta(\nu, 1)$, $\eta(\nu, 10^6)$ were calculated and these were examined to see whether $\eta(\nu, N)$ was equal to $\frac{c}{\omega} KN$, i.e. if $\eta(\nu, N) = KN$, then $\eta(\nu, 1) = K$, so that for any fixed frequency $\eta(\nu, N) = \eta(\nu, 1)N$ provided $\eta(\nu, N) = KN$.

In general, one may formally define a function $\mathcal{K}$ such that $\eta = \mathcal{K}N$, where $\mathcal{K} = \mathcal{K}(\nu, \omega, N)$. Under certain circumstances $\mathcal{K}$ should be identical to K; it is these circumstances which were started to be investigated at the end of the coop period.

Some of the curves obtained can be seen in Appendix H. They cover a wide range of frequencies: 80, 40, 20, 10, 5, 3, 2.5 and 1.5 mc/s for the angle of propagation θ = 1.27 radians.

C - Frequency Dependence of Absorption

It has been indicated previously that because no analytical profiles existed for N and ν the absorption had to be calculated numerically. The plots presented in Appendix F are for η_x and η_o but these are merely the respective integrands for ordinary and extraordinary absorption divided by $\frac{\omega}{c}$. Since the present paper is only interested in the absorption undergone by a wave which comes from outer space, the absorption integrals were calculated only for those frequencies which could penetrate the ionosphere under the assumed conditions.

These integrations were done using what amounts essentially to the trapezoid rule. This rule of integration, according to books in numerical analysis, gives, for sufficiently small size of the interval h of integration, that

$$\int_{x_o}^{x_o+h} y\,dx = h\left[y_o + \tfrac{1}{2}\Delta y_o\right] = h\left[\frac{y_o}{2} + \frac{y_1}{2}\right] \qquad (5.1)$$

Where $\Delta y_o = y(x_o+h) - y(x_o)$. In this paper the integration was merely approximately by a sum; thus:

$$\int \eta\,dh \simeq \Delta h \left(\sum_i \eta_i\right) \qquad (5.2)$$

Where the sum is carried between the appropriate limits and $\Delta h = 5$ km. From (5.1) above, it may be seen that when $\int_{x_0}^{x_0+h} y\,dx$ is extended to a large range for dx then $\int y\,dx = h\left(\sum_{i=1}^{n-1} y_i + \frac{y_0}{2} + \frac{y_n}{2}\right)$ The difference between this and (5.2) is merely $\frac{y_0}{2} + \frac{y_n}{2}$ which in the present case is $\frac{f_0}{2} + \frac{f_n}{2}$ and this is very small for any frequency judging by the profiles presented in Appendices F and G.

The trapezoidal rule itself is a curde approximation, but first, Δh may be considered to be small enough, this is strikingly demonstrated by the graphs in Appendix G, and second, the main interest in performing these integrations is in bringing out the frequency dependence of absorption, it is therefore felt, that given the inaccuracies present in the data itself any more specialized rule such as Simpson's rule will not bring out any new fact about the frequency dependence.

It was estimated that the contributions to the integral from heights above 700 km were less than 1% of the total integral, hence the integrals were carried out to 700 km.

It should, at this point, be emphasized that these integrals include deviative absorption that might be present since no particular assumption was made about μ , although no attempt was made to separate the contributions from both components, and, as was mentioned previously, no attempt was made either to estimate rigorously the path of the waves, something which constitutes a problem unto itself.

As a matter of convenience the integrals were calculated up to 375 km and from 375 km upward for night-time conditions. This made it possible to talk about contributions from different parts of the

ionosphere to the total absorption. For the day-time, the following segments were integrated separately: up to 105 km, from 105 to 200 km, 200-310 km, and 310-700 km. Each of these regions includes a certain ionospheric region of interest as may be seen, from the plots in Appendix G. The results are presented below, where A_o denotes ordinary absorption and A_x extraordinary (in the sense of the magneto-ionic components):

Table I

Night-Time Absorption in db for Several Frequencies

f (mc/s)	$A_o(<375)$	$A_o(>375)$	Total	$A_x(<375)$	$A_x(>375)$	Total
80	0.047	.115	.162			
40	0.2	0.46	.66			
20	0.85	1.9	2.75	0.91	1.98	2.89
14	1.53	3.9	5.43	1.8	4.35	6.15
10	3.6	7.8	11.4	4.2	9.1	13.3
7	7.2	15.7	22.9	12.0	24.7	36.7

Table II

Day-Time Ordinary Absorption in db for Several Frequencies

f(mc/s)	$A_o(<105)$	$A_o(105-200)$	Sub Total	$A_o(200-310)$	$A_o(310-700)$	Sub Total	Total
80	0.25	0.16	.42	0.25	0.17	.42	.84
40	1.01	0.76	1.77	1.01	0.59	1.60	3.37
20	3.78	3.02	6.80	4.34	3.33	7.67	14.47
14	7.37	6.06	13.43	10.08	7.7	17.78	31.21
10	14.3	12.2	26.5	42.5	15.7	58.2	84.7

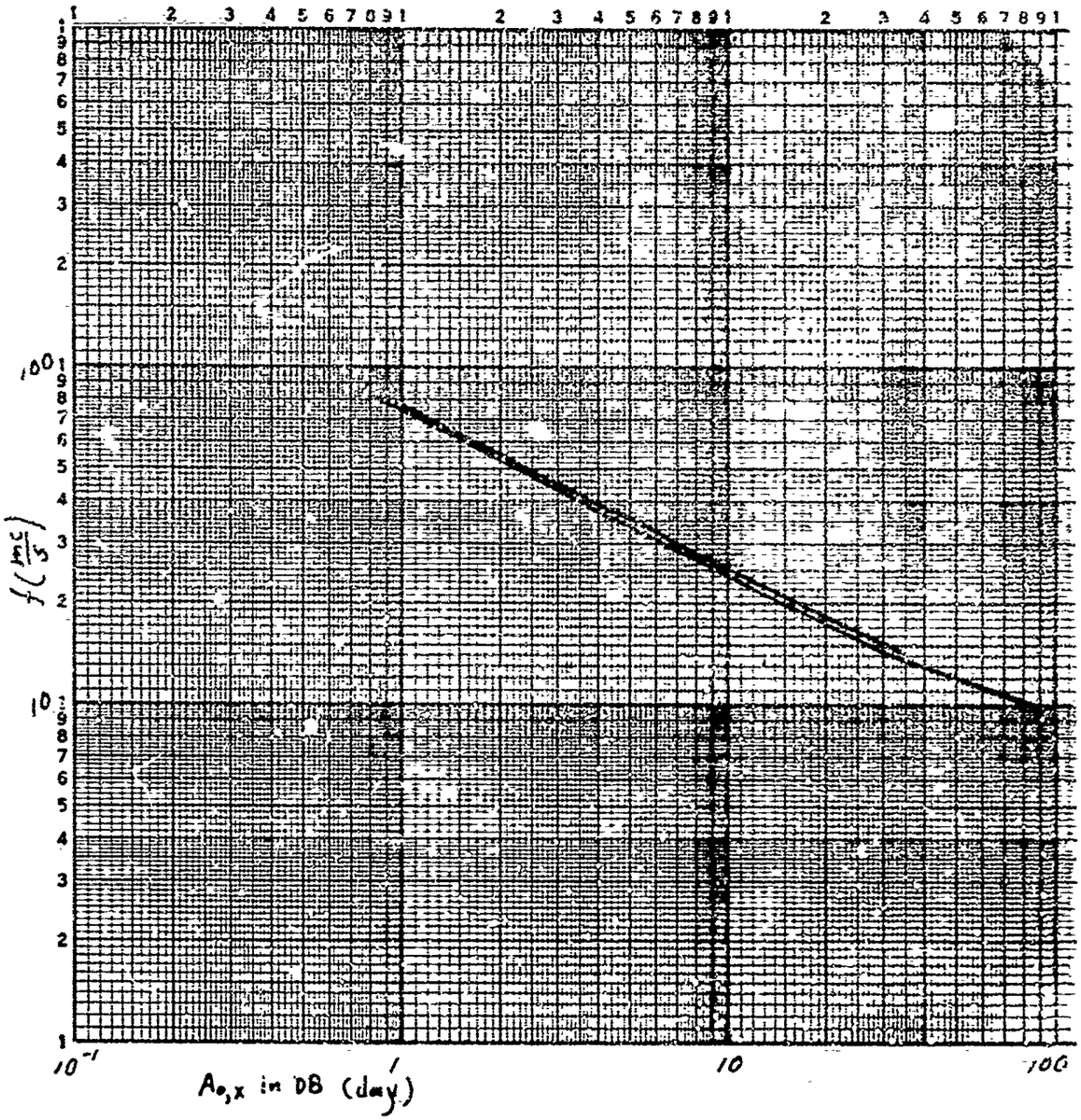
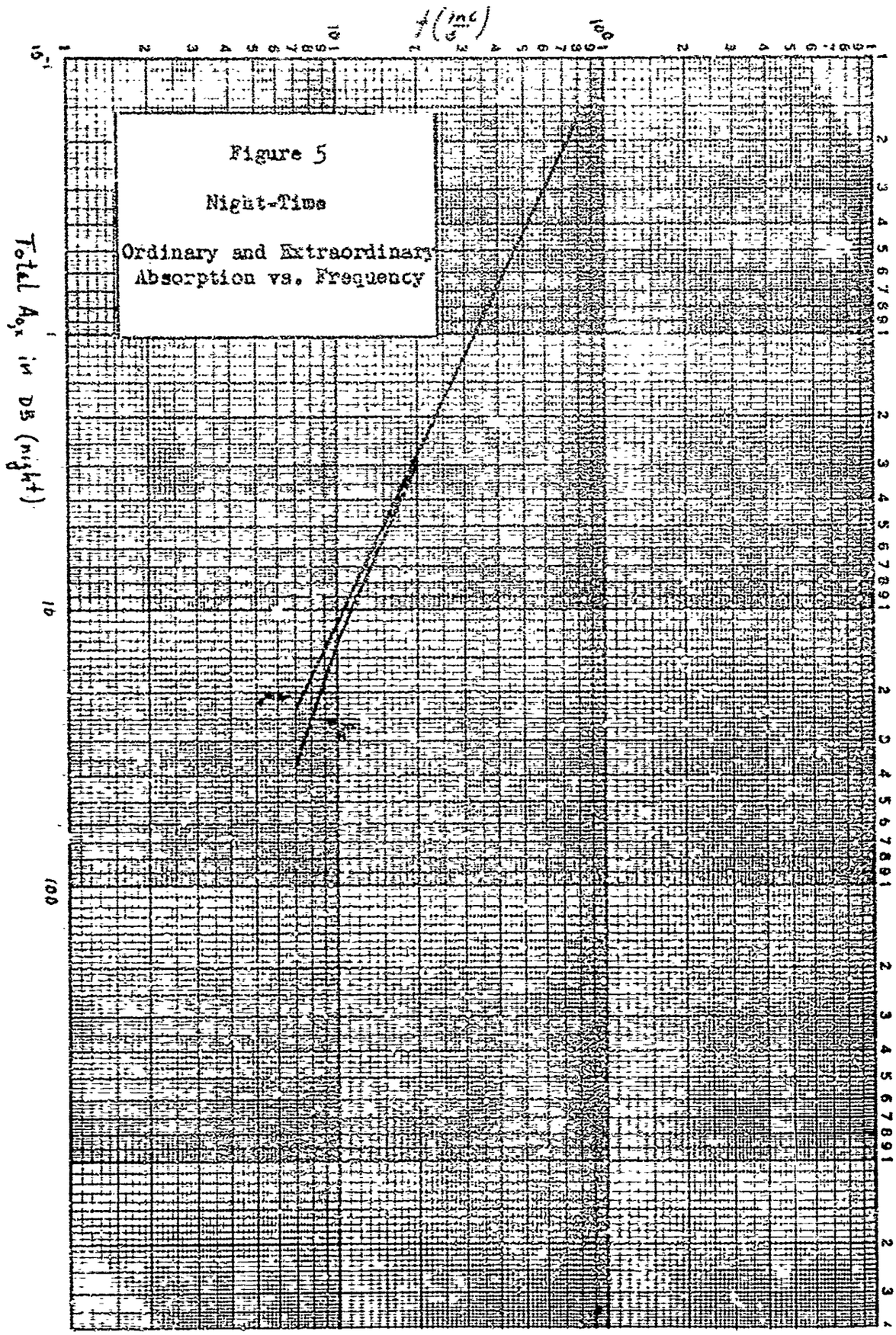


Figure 3

Day-Time

Ordinary and Extraordinary
Absorption vs. Frequency

Figure 4
Day-Time
Differential Absorption
vs. Frequency
Differential Absorption in DB/(day)
f (m/s)


f (inc)
Total A_{ox} in DB (night)
Figure 5
Night-Time
Ordinary and Extraordinary
Absorption vs. Frequency

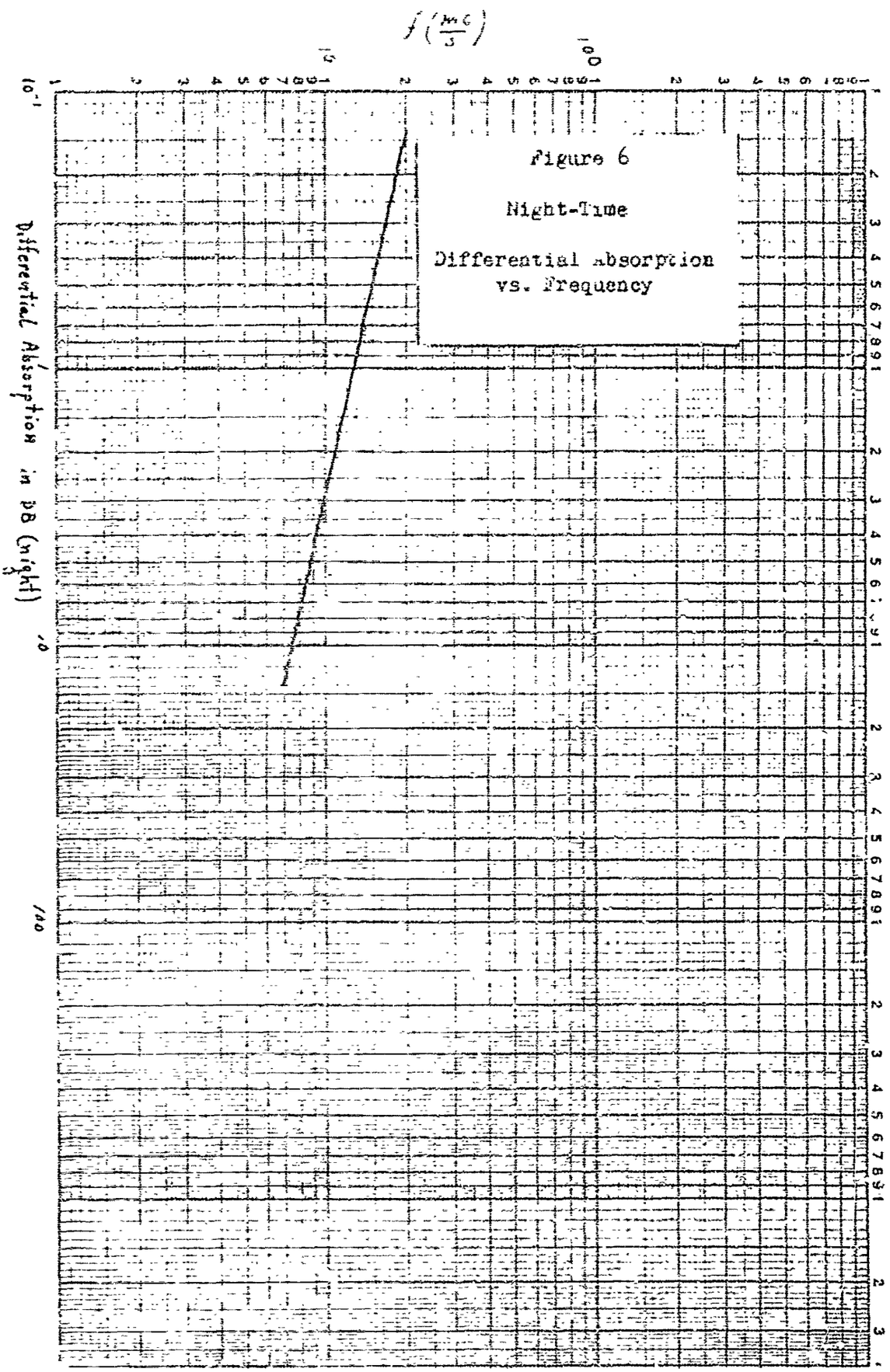


Figure 6

Night-Time

Differential Absorption
vs. Frequency

Table III

Day-Time Extraordinary Absorption in db for Several Frequencies

$f(\frac{mc}{s})$	$A_x(>105)$	$A_x(105\text{-}200)$	$A_x(200\text{-}310)$	$A_x(310\text{-}700)$	Total
80	.26	.19	.25	.20	.90
40	1.05	.80	1.06	.79	3.70
20	4.08	3.32	4.72	3.58	15.70
14	8.19	6.18	11.63	8.19	34.19

The frequency dependence of absorption may be demonstrated visually by plots of the data presented in the Tables. Also, plots for differential absorption have been included (See Figures 3, 4, 5, 6).

The determination of polynominal relations is relatively easy, though labourious, once given a set of points, but these formulas would obviously be merely a best fit for the data and would have to be changed any time a new point is added to the data, so for the curves given in the log-log graphs, no attempt has been made of doing this.

Yet, it can be seen that the plot for total absorption for the ordinary ray at night is a fairly straight line. So that $\log A = -n \log f$. A determination of this n leads to $n = 2.0$. For the extraordinary ray, the first three points form also approximately a straight line with a slope of $n = 1.4$. For differential absorption, although there is more spread in the points, a straight line can also be drawn to connect the points. The slope of this line is $n = 4.3$. For day-time conditions, no such determination is possible, for the curve joining the points has a slight curvature. In this case, A is a more complicated function of

frequency than $A = \frac{a}{f^n}$.

Therefore, for night-time conditions as assumed here, $A_o = \frac{a_o}{f^2}$ and $A_x = \frac{a_x}{f^{1.4}}$, in the most general forms. That the exponents are different was to be expected since differential absorption goes to o, or at least approaches undetectable differences, when the frequency increases. This is confirmed by the fact that the slope determined for differential absorption was $n = 4.3$, which means that it decreases faster than either ordinary or extraordinary absorption.

The situation is somewhat more complicated under day-time conditions. The curve for ordinary and extraordinary absorption are not straight lines because of the deviation of the lower points from a straight line. This is due, no doubt, to the presence of deviative absorption which for longitudinal propagation does not follow the inverse square law for the frequency and would be expected to depart under general conditions from any such simple behaviour. In fact, the upper points of ordinary absorption for day-time conditions also follow an inverse square law. The differential absorption follows a straight line despite some scatter in points with an $n = 2.2$, which shows a slower decrease with frequency than that occuring under night-time conditions.

It should also be noted that the ratio of the absorption at night below 375 km to that over 375 km is fairly constant for all frequencies for both the ordinary and the extraordinary ray (See Figure 7). This ratio is about .44. It was expected that it would be constant, given the form for absorption $A_{o,x} = \frac{a_{o,x}}{f^2}$. It is also seen that this ratio is

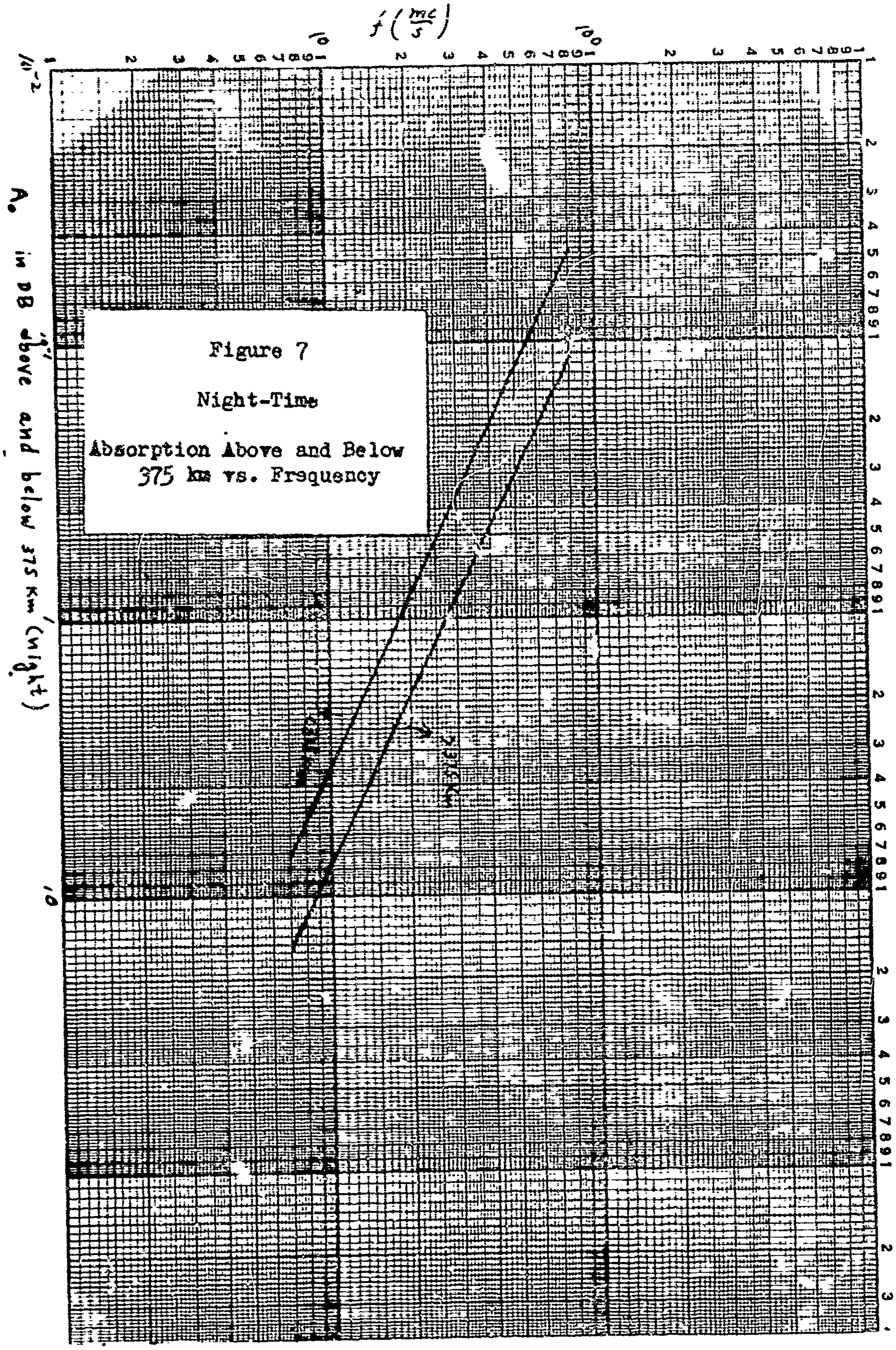


Figure 7

Night-Time

Absorption Above and Below 375 km vs. Frequency

about the same as the ratio of the electron content below 375 km to that above 375 km, an observation which is confirmed by Faraday rotation determinations of the electron content in the ionosphere.

D - On the Height Profiles of μ and γ

The day-time attenuation curves show that there are three main absorption components, one at about 300 km and the other two at approximately 80 km and 105 km. The upper peak corresponds to F-region absorption, the other two to D and E region absorption. These components were detected by Mitra and Shain (Ref 35) in measurements of ionospheric absorption at 18.3 mc/s, although they separate these components into only two: F-region absorption and the rest of the absorption which they say is mainly D-region absorption. The results presented in Appendix G and Table II suggest that E-region absorption is also important.

For night-time absorption, there is only a broad maximum of absorption corresponding to the F-layer.

At high altitudes the difference between γ_x and γ_o becomes nearly constant. It was shown in previous chapters, that for longitudinal propagation and high frequencies, this was to be expected. The fact that it also occurs for a case of general propagation suggests that it has a more general range of validity. If $\log \gamma_x - \log \gamma_o$ is constant, then $\gamma_x = C_1 \gamma_o$ or $K_x = C_1 K_o$, so that for ranges where this has been determined to be true only one integration need be performed to calculate absorption.

At low frequencies (less than the critical frequency of the layer) the behaviour of μ and η is very complicated. Absorption for these frequencies could also be determined but they require more advanced methods of computation based on contour integrals. (See Budden, Ch. 20).

The μ profiles suggest that under certain conditions waves may be trapped at a certain height. For example, at the conditions assumed in this paper for day-time, a wave of frequency 10 mc/s and ordinary polarization, if it orginates at an altitude of about 300 km on an oblique path, will, on going to higher altitudes, find a higher value of μ and hence will be refracted down, on traveling down, however, it will again encounter higher values of μ and hence will be refracted upward. It will stay thus within the limits imposed by the μ profile until it is absorbed.

Thus by these brief comments, one may see that the height profiles of μ and η alone merit some interest. These curves have been investigated extensively as a function of N and ν, not of h, in Budden's and Ratcliffe's book.

It may be added that the profiles presented in this paper were those computed by using the Sen-Wyller magneto-ionic formula, but that comparison computations were made using the classical Appleton-Hartree formula. It was mentioned previously that Shkarofsky had showed that the classical Appleton-Hartree equation could be used in the high ionosphere provided one used the appropriate collision frequency. It is

also shown in the Sen-Wyller paper that in the limit $\nu < .1\omega$ the Appleton-Hartree could be retained provided one used a collision ν frequency ν_{AH} equal to 5/2 ν_m (that used in the Sen-Wyller formula) and in the limit $\nu >> \omega$ the Appleton-Hartree could be retained if ν_{AH} was set equal to 3/2 ν_m . It was found that $\kappa_{SW} = 2.5 \kappa_{AH}$ for all the height range, where κ_{SW} and κ_{AH} are the absorption coefficients as computed by the Sen-Wyller and the Appleton-Hartree formulations respectively. If the Sen-Wyller coefficients are used to compute the total absorption then a certain error results which, using Shkarofsky's remarks and the results mentioned above, may be easily seen to be $E = 1.5 \int_{F_2}^{\infty} \kappa_{AH}\, dh$ where $\int_{F_2}^{\infty} \kappa_{SW}\, dh$ is the absorption as computed from the Appleton-Hartree formulation in the F_2 region, which may be assumed to be from 310 km up. That is, the $'$ error would be $\frac{1}{1.6} \int_{F_2}^{\infty} \kappa_{SW}\, dh$ where $\int_{F_2}^{\infty} \kappa_{SW}\, dh$ is the F_2 region absorption as computed from the Sen-Wyller formulation. This error is an over-estimation of the absorption since in the F_2 region absorption coefficients which are 2.5 higher than the more appropriate ones, then $\log(A+E) - \log A$ represents the shift in log coordinates where A is the correct absorption, or $\log P - \log(P-E)$ where P is the plotted absorption. But $\log P - \log(P-E) = -\log\left(1 - \frac{E}{P}\right)$ where $\frac{E}{P} << 1$. Hence

$$\log P - \log(P-E) \approx -\left(-\frac{E}{P}\right) = \frac{E}{P}$$

where terms of order two or higher have been neglected. This ratio is not only small, .13 for extraordinary day-time absorption and .12 for the ordinary on the average, but, perhaps surprisingly independent of

frequency, hence, the expressions determined for frequency dependence are not affected because the shift is constant for each ordinate. It therefore, appears to justify the use of Sen-Wyller only, although, it is realized that in some other future study both should be used.

The fact that $\kappa_{SW} = 2.5\ \kappa_{AH}$ in the present computations is by no means a general relationship but the conditions under which it is obtained should be investigated. Time precluded its investigation in the present paper. Because in most of the ionosphere $\nu < \omega$ for the present computations, then if $\nu_{AH} = \frac{5}{2}\nu_m$ as mentioned previously, the Appleton-Hartree equation can be retained; in general cases, it is not readily possible to say what this implies because of the complicated algebraic formula but in a simple and useful case one can derive certain requirements, i.e. the longitudinal case: in that case and for $\mu \sim 1$, κ_{AH} written in terms of the frequencies is:

$$\frac{\omega_0^2 \nu_{AH}}{2\omega\left[(\omega+s)^2 + \nu_{AH}^2\right]}$$

while κ_{SW} is

$$\frac{5\omega_0^2}{4\omega\nu_m} P_{5/2}\left(\frac{\omega+s}{\nu_m}\right)$$

then if $\nu_{AH} = \frac{5}{2}\nu_m$

$$\kappa_{AH} = \frac{5\omega_0^2 \nu_m}{4\omega\left[(\omega+s)^2 + \frac{25}{4}\nu_m^2\right]}$$

and since $\nu_m^2 \ll \omega^2$, $\kappa_{AH} \simeq \frac{5\omega_0^2\nu_m}{4\omega\nu_m^2\left[\frac{25}{4}\right]} = \frac{5\omega_0^2}{4\omega\nu_m}\left(\frac{4}{25}\right)$

so that $\rho_{5/2}\left(\frac{\omega+s}{\nu_m}\right) \sim \frac{4}{25}$ within the limits of error obtained by Sen and Wyller (Ref 49). If ν_{AH} were ν_m as was assumed for the Appleton-Hartree computations that were made, then

$$\kappa_{AH} = \frac{\omega_0^2}{2\omega\nu_m\left[\left(\frac{\omega+s}{\nu_m}\right)^2+1\right]}$$

and $5/2\ \kappa_{AH} = \kappa_{SW}$ then

$$\kappa_{SW} = \frac{5\omega_0^2}{4\omega\nu_m\left[\left(\frac{\omega+s}{\nu_m}\right)^2+1\right]}$$

so that $\kappa_{SW} = 2.5\,\kappa_{AH}$ would imply for longitudinal propagation that $\rho_{5/2}\left(\frac{\omega+s}{\nu_m}\right) \sim \frac{1}{\left(\frac{\omega+s}{\nu_m}\right)^2+1}$

E - On K or Absorption Per Electron

Some aspects of K may be explained qualitatively if use is made of several results developed in Chapter 3, Equation (3.24) gave K as a function of frequency and it was shown that K has a maximum at $\nu=\omega_c$, and it was therefore explained that this result implied that the maximum for K was higher for the extraordinary ray than for the ordinary. The graphs in Appendix H show the same type of variation. That is, their maximum is frequency dependent and the higher the frequency the lower the maximum; also K_x is generally higher than K_o although for low frequencies the behaviour of both of these becomes erratic and they cross over.

The main question asked about κ was whether it was proportional to N or not, i.e. whether $\kappa = KN$ where K depends only on the frequency and the collision frequency profile. The computed results show that for the Sen-Wyller approach (and hence for the Appleton-Hartree formulation

according to the comments made before) and under the conditions assumed in this specific problem (dip angle of 73° and gyrofrequency of 1.57 mc/s corresponding to Sagamore Hill Observatory, Mass. and the assumed profiles) the linearity condition for the ordinary ray holds within certain limits. It is good enough for frequencies above 2 mc/s at low heights where the collision frequency is high and the electron density is low. As N increases the linearity condition becomes less valid. It may be said that the linearity condition may be taken to be valid up to about for frequencies down to 2 mc/s and up to 10^6 for frequencies down to 10 mc/s. For high frequencies such as 10, 40, and 80 mc/s it may be taken to be valid at densities of 10^6 /cc which are densities which occur in Polar Cap Absorption events. Outside these limits deviations from linearity may be considerable.

As a side result, it may be noted that the height at which differential absorption appears is dependent on the collision frequency and the frequency of the wave, so that if a rocket probe with a transmitter at a certain frequency wave sent into the ionosphere, then that height at which the powers received in the two wave modes begin to differ appreciably is characteristic of a certain collision frequency. If the transmitter were a multifrequency one, some mapping of the collision frequency profile at low heights could be done. The idea is worth considering.

A Method for Measuring Cosmic Noise Levels and Total Absorption

It has been mentioned before that the $\log K_x - \log K_o$ was largely

especially, in the high ionosphere as was shown by the plots in Appendix G. This implies that, since $\kappa_x = C_1 \kappa_o$, then

$$\frac{A_x - A_o}{A_o} = C_1 - 1$$

Where the constant C_1 depends only on frequency. Therefore, this constant once theoretically known permits the assertion that $D = \psi(f) A_o$ where $\psi(f)$ is a percentage. Now, since D is measurable by riometers, then A_o can be directly calculated and hence the cosmic noise level. By way of illustration, if $\psi(f) = .1$ and it is determined that $D = 10\,DB$, then A_o is 10 DB and from $A_o = P - P_o$, where P is the cosmic noise level and P_o the power receiver in the ordinary component above a certain reference level, then $P = 10 + P_o$. This method is close to that of Little, etal (Ref). Although they determine P by extrapolation of a plot of P_o vs. $P_o - P_x$. For more details, the original paper should be consulted.

In the Appleton-Hartree formulation and for longitudinal propagation, for non-deviative absorption, $C_1 = \left(\frac{\omega_{e_1}}{\omega_{e_2}}\right)^2$, so that it is very simple to compute $\psi(f)$ for any frequency. It is of interest to see whether propagation in any general direction would alter this situation very much. In the case presented here, Appendix C and A tend to indicate that it does not, for graph 3 in Appendix H, indicate that for a given N , the differential absorption once it appears is largely insensitive to the collision frequency since it remains constant for the heights examined, Appendix G, as mentioned before, shows $\log \kappa_x - \log \kappa_o$ for the high

ionosphere is constant. Lastly, so does the linearity of γ at low heights. This fact makes the selection of the profiles explained in Chapter IV less arbitrary than would seem.

VI. CONCLUSIONS AND RECOMMENDATIONS

The main point that this paper tries to make is that differential absorption may serve to calculate total absorption and cosmic noise levels. For example, differential absorption is directly measurable and if theoretically one estimates that for a certain frequency the differential absorption represents 10% of the extraordinary absorption then for, say, $1\,DB$ differential absorption, the extraordinary absorption for such a frequency is $10\,DB$, then as mentioned previously, the cosmic noise level is the sum of $10\,DB$ and the power in DB above a certain reference level received in the extraordinary mode.

Also, the frequency dependence of absorption A_o, A_x and differential absorption was determined to be: $\frac{a_o}{f^2}$, $\frac{a_x}{f^{1.4}}$, $\frac{c}{f^{4.3}}$, respectively at night-time. During day-time conditions, no formula appears to fit the data, but the upper points of the ordinary absorption curve appears to obey the law $\frac{a_o}{f^2}$, differential absorption was proportional to $\frac{1}{f^{2.2}}$.

The approximation $f=KN$ used often in ionospheric physics is showed to be valid under general conditions at low heights and high frequencies and for low N. Outside those limits considerable departures can occur.

Recommendations:

It is recommended that a systematic study be undertaken of the variations undergone by $\frac{A_x - A_o}{A_o}$ as the angle of propagation varies for different typical profiles, with the intention of telling how much independence from the profiles actually exist and how they affect the percentage differential absorption. Furthermore, the absorption calculations should be combined with a ray path analysis to take into account deviative losses with more accuracy.

An experiment should be performed after better calculations are done of $\frac{A_x - A_o}{\mu_a}$ to determine total absorption from percent differential absorption and also cosmic noise levels outside the ionosphere. Lastly, rocket measurements as low frequencies could be used to determine collision frequency profiles.

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APPENDIX A

The Wave Magnetic Field

A wave traveling in the Z direction and linearly polarized with its electric vector in the x-direction, will produce a force on an electron in the x-direction

$$m \dot{v}_x = e E_0 \cos \omega t \tag{A.1}$$

$$v_x = \frac{e E_0}{m \omega} \sin \omega t \tag{A.2}$$

from equation (1.15), and taking $\underline{\kappa} = \frac{\omega}{c} M \underline{n}$, where M is the refractive index (in general, complex) and $\underline{n}$ is a unit vector in the direction of propagation, Z , ($\underline{n} = \hat{\kappa}$, c is the velocity of light, the following is obtained:

$$\frac{\omega}{c} M |\hat{\kappa} \times \underline{E}_x| = \mu_0 \omega H_y$$

$$M E_x = \mu_0 c H_y$$

or

$$H_y = \frac{M E_x}{\sqrt{\frac{\mu_0}{\epsilon_0}}}$$

Therefore, the magnetic force experienced by the electron in the direction y is:

$$e \mu_0 H_y v = e \mu_0 \left(\frac{e E_0}{m \omega}\right) \sin \omega t \left(\frac{M E_0}{\sqrt{\mu_0/\epsilon_0}}\right) \cos \omega t$$

$$= \frac{e^2 E_0^2 M}{m \omega c} \sin \omega t \cos \omega t$$

This force is a maximum when $\sin\omega t \cos\omega t = 1/2$

$$\therefore F_M = \frac{e^2 E_0^2 M}{2m\omega c}$$

The ratio of the magnetic force to the maximum electric force, F_M/F_e is:

$$F_M/F_e = \frac{e^2 E_0^2 M}{2m\omega c} / cE_0$$

$$= \frac{eE_0 M}{2m\omega c}$$

"The value of E_0 at 100 Km from a transmitter radiating 10^6W is of the order 0.08 V/M, and the magnetic field of the wave is about 2.7×10^{-6} oersted. The magnetic and electric fields encountered in the ionosphere from man-made radio transmitters will rarely reach these values, and the frequency is always greate· 10 kc/s. The refractive index is of the order of unity. Hence the ratio will not exceed 3.7×10^{-4}". As can be seen this is a small ratio in the "worst" of circumstances. This justifies it being neglected in the equations of motion.

APPENDIX B

Determination of the Appleton-Hartree Formula

The dispersion relation (111) in the Appleton-Hartree formulation is:

$$\frac{\sin^2\theta}{\frac{1}{\epsilon_1}-\frac{1}{M^2}} + \frac{\sin^2\theta}{\frac{1}{\epsilon_2}-\frac{1}{M^2}} + \frac{2\cos^2\theta}{\frac{1}{\epsilon_3}-\frac{1}{M^2}} = 0 \qquad (111)$$

where

$$\epsilon_1 = 1 - \frac{a}{\beta+\Omega} = 1 - \frac{X}{1+Y-iZ} \qquad (B.1)$$

$$\epsilon_2 = 1 - \frac{a}{\beta-\Omega} = 1 - \frac{X}{1-Y-iZ} \qquad (B.2)$$

$$\epsilon_3 = 1 - \frac{a}{\beta} = 1 - \frac{X}{1-iZ} \qquad (B.3)$$

Equation (111) may be recast in the form

$$\frac{\epsilon_1\sin^2\theta}{M^2-\epsilon_1} + \frac{\epsilon_2\sin^2\theta}{M^2-\epsilon_2} + \frac{2\epsilon_3\cos^2\theta}{M^2-\epsilon_3} = 0 \qquad (B.4)$$

The summation of these leads to

$$M^4[(\epsilon_1+\epsilon_2)\sin^2\theta + 2\epsilon_3\cos^2\theta] + M^2[\epsilon_1\sin^2\theta(\epsilon_3-\epsilon_2) + \epsilon_2\sin^2\theta(\epsilon_3-\epsilon_1) - 2\epsilon_3(\epsilon_1+\epsilon_2)] + 2\epsilon_1\epsilon_2\epsilon_3 = 0 \qquad (B.5)$$

A series of redefinitions is useful at this point:

$$A = \epsilon_1 + \epsilon_2 \tag{B.6}$$

$$B = 2\epsilon_3 \tag{B.7}$$

$$C = \epsilon_1(\epsilon_3 - \epsilon_2) + \epsilon_2(\epsilon_3 - \epsilon_1) \tag{B.8}$$

$$D = BA \tag{B.9}$$

$$E = 2\epsilon_1\epsilon_2\epsilon_3 \tag{B.10}$$

so that equation (B.5) then becomes:

$$M^4[A\sin^2\theta + B\cos^2\theta] + M^2[\sin^2\theta C - D] + E = 0 \tag{B.11}$$

If use is made of the formula for determining the solutions for a quadratic equation, then

$$M^2 = 1 - \frac{2[(A\sin^2\theta + B\cos^2\theta) + C\sin^2\theta - D] + E}{2(A\sin^2\theta + B\cos^2\theta) - D \pm \sqrt{(C\sin^2\theta - D)^2 - 4(A\sin^2\theta + B\cos^2\theta)E}} \tag{B.12}$$

The terms inside the radical may be seen to be:

$$C^2\sin^4\theta - 2CD + D^2 - 4AE + \cos^2\theta(2CD + 4AE - 4BE) \tag{B.13}$$

But by the definitions of A, B, C, D, E

$$D^2 - 2CD - 4AE = 0 \tag{B.14}$$

and $2CD+4AE-4BE$ may be seen to be $4(\epsilon_3)^2(\epsilon_1-\epsilon_2)^2$. This last expression is set equal to F^2, so that the terms inside the radical become

$$C^2\sin^4\theta + F^2\cos^2\theta \tag{B.15}$$

The use of $\mathbb{K}_1$, $\mathbb{K}_2$ and $\mathbb{K}_3$ is no longer useful, so from here on the ionospheric parameters X, Y, Z are used. With this in mind,

$$\frac{F^2}{4}\cos^2\theta = \left(\frac{1-X-iZ}{1-iZ}\right)^2\left[\frac{2X}{(1-Y-iZ)(1+Y-iZ)}\right]^2 Y_L^2 \qquad \text{(B.16)}$$

Also

$$\frac{c^2\sin^4\theta}{4} = \frac{\left[\frac{2X}{(1-iZ)(1-Y-iZ)(1+Y-iZ)}\right]^2 Y_T^4}{4} \qquad \text{(B.17)}$$

So that the terms inside the radical divided by 4 can be expressed as

$$\left[\frac{1-X-iZ}{1-iZ}\right]^2\left[\frac{2X}{(1-Y-iZ)(1+Y-iZ)}\right]^2\left\{\frac{Y_T^4}{4(1-X-iZ)^2}+Y_L^2\right\} \qquad \text{(B.18)}$$

The term in the numerator of the fraction may be written as:

$$\frac{2(1-iZ-X)}{(1-iZ-Y)(1-iZ+Y)(1-iZ)}[X^2] \qquad \text{(B.19)}$$

and the other term in the denominator is

$$\left[\frac{XY^2}{(1-Y-iZ)(1+Y-iZ)(1-iZ)}\right]\sin^2\theta + \frac{2X(1-iZ-X)}{(1-Y-iZ)(1+Y-iZ)} \qquad \text{(B.20)}$$

So that, finally,

$$M^2 = 1 - \frac{\frac{2(1-iZ-X)K^2}{(1-iZ-Y)(1-iZ+Y)(1-iZ)}}{\frac{2X(1-iZ-X)}{(1-iZ-Y)(1+Y-iZ)} - \frac{XY^2\sin^2\phi}{(1-iZ)(1-Y-iZ)(1+Y-iZ)} \pm \left\{\frac{1-Y-iZ}{1-iZ}\left[\frac{2X}{(1-Y-iZ)(1+Y-iZ)}\right]^2\left[\frac{Y_T^4}{4(1-X-iZ)^2}+Y_L^2\right]\right\}^{\frac{1}{2}}} \qquad \text{(B.21)}$$

It follows immediately from (B.21) that

$$M^2 = 1 - \frac{X}{1 - iZ - \frac{Y_T^2}{2(1-X-iZ)} \pm \sqrt{\frac{Y_T^4}{4(1-X-iZ)^2} + Y_L^2}}$$

This is the Appleton-Hartree equation.

APPENDIX C

Determination of $\alpha_2, \beta_2, \xi_2, \eta_2, \gamma_2, \delta_2$

Boltzmann's equation becomes, when reduced by the simplifications made in Chapter II:

$$\frac{\partial f_2}{\partial t} + \left[\underline{\Gamma}_2 \cos\omega t + \frac{e_2}{m_2}(\underline{V}_2 \times \underline{H}_0)\right] \cdot \nabla_{V_2} f_2 = \iiint (f_1' f_2' - f_1 f_2) g b db d\epsilon d\underline{V},$$

Furthermore in the Chapman-Enskog method of solution

$$f_2 = f_2^{(0)} + \underline{\Gamma}_2 \cdot \underline{V}_2 (\alpha_2 \cos\omega t + \beta_2 \sin\omega t) + (\underline{H}_0 \times \underline{\Gamma}_2) \cdot \underline{V}_2 (\xi_2 \cos\omega t + \eta_2 \sin\omega t)$$

$$+ [\underline{H}_0 \times (\underline{H}_0 \times \underline{\Gamma}_2)] \cdot \underline{V}_2 (\gamma_2 \cos\omega t + \delta_2 \sin\omega t) \qquad (2.7)$$

In principle one substitutes equation (2.7) in Boltzmann's equation and equates the coefficients of Γ_2^2, $\underline{\Gamma}_2 \cdot \underline{V}_2$, $(\underline{H}_0 \times \underline{\Gamma}_2) \cdot \underline{V}_2$ and $[\underline{H}_0 \times (\underline{H}_0 \times \underline{\Gamma}_2)] \cdot \underline{V}_2$ on both sides of the equation to each other. It is easier to do it term by term instead of just outright substitution.

It should be remembered that for any function f(r),

$$\nabla_r f(r) = \frac{\partial f}{\partial r} \nabla_r r \qquad (C.1)$$

Therefore

$$\nabla_{V_2} f_2 = \frac{\partial f_2}{\partial V_2} \left(\frac{\underline{V}_2}{V_2}\right) \qquad (C.2)$$

as a help in differentiating dots products of $\underline{V}_2$ with some other vector, $\nabla_{V_2} f_2$ can be written also as $\frac{\partial f_2}{\partial \underline{V}_2}$ which merely means that $\frac{\partial}{\partial \underline{V}_2}$ is the vector operator whose ith component is $\frac{\partial}{\partial V_{2i}}$, so that when it

operates on a dot product $A_j v_{2_j}$ where the j's imply summation, then $\frac{\partial}{\partial v_{2_i}}(A_j v_{2_j}) = A_i$. With these considerations in mind then

$$\nabla_{v_2} f_2 = \frac{v_2}{v_2}\left\{\frac{\partial f_2^{(0)}}{\partial v_2} + \underline{\Gamma}_2 \cdot \underline{v}_2 \left(\frac{\partial \alpha_2}{\partial v_2}\cos\omega t + \frac{\partial \beta_2}{\partial v_2}\sin\omega t\right)\right\} + (\alpha_2 \cos\omega t + \beta_2 \sin\omega t)\underline{\Gamma}_2$$

$$+ \frac{v_2}{v_2}\left\{\left(\frac{\partial \xi_2}{\partial v_2}\cos\omega t + \frac{\partial \eta_2}{\partial v_2}\sin\omega t\right)(\underline{H}_0 \times \underline{\Gamma}_2)\cdot \underline{v}_2\right\} + (\xi_2 \cos\omega t + \eta_2 \sin\omega t)(\underline{H}_0 \times \underline{\Gamma}_2)$$

$$+ \frac{v_2}{v_2}\left\{\left(\frac{\partial \gamma_2}{\partial v_2}\cos\omega t + \frac{\partial \delta_2}{\partial v_2}\sin\omega t\right)[\underline{H}_0 \times (\underline{H}_0 \times \underline{\Gamma}_2)]\cdot \underline{v}_2\right\} + (\gamma_2 \cos\omega t$$

$$+ \delta_2 \sin\omega t)[\underline{H}_0 \times (\underline{H}_0 \times \underline{\Gamma}_2)] \qquad (C.3)$$

Therefore

$$\underline{\Gamma}_2 \cos\omega t \cdot \nabla_{v_2} f_2 = \frac{\underline{\Gamma}_2 \cdot \underline{v}_2 \cos\omega t}{v_2}\left\{\frac{\partial f_2^{(0)}}{\partial v_2} + \underline{\Gamma}_2 \cdot \underline{v}_2\left(\frac{\partial \alpha_2}{\partial v_2}\cos\omega t + \frac{\partial \beta_2}{\partial v_2}\sin\omega t\right)\right.$$

$$+ \left(\frac{\partial \xi_2}{\partial v_2}\cos\omega t + \frac{\partial \eta_2}{\partial v_2}\sin\omega t\right)(\underline{H}_0 \times \underline{\Gamma}_2)\cdot \underline{v}_2 + \left(\frac{\partial \gamma_2}{\partial v_2}\cos\omega t + \frac{\partial \delta_2}{\partial v_2}\sin\omega t\right)$$

$$\left.\times[(\underline{H}_0) \times (\underline{H}_0 \times \underline{\Gamma}_2)\cdot \underline{v}_2\right\} + \Gamma_2^2 \cos\omega t(\alpha_2 \cos\omega t + \beta_2 \sin\omega t) -$$

$$(\gamma_2 \cos\omega t + \delta_2 \sin\omega t) H_0^2 \Gamma_2^2 \sin^2\psi \cos\omega t \qquad (C.4)$$

and also

$$\frac{e_2}{m_2}(\underline{v}_2 \times \underline{H}_0)\cdot \nabla_{v_2} f_2 = \frac{e_2}{m_2}(\alpha_2 \cos\omega t + \beta_2 \sin\omega t)(\underline{H}_0 \times \underline{\Gamma}_2)\cdot \underline{v}_2 +$$

$$\frac{e_2}{m_2}(\xi_2 \cos\omega t + \eta_2 \sin\omega t)[\underline{H}_0 \times (\underline{H}_0 \times \underline{\Gamma}_2)]\cdot \underline{v}_2 -$$

$$\frac{e_2}{m_2} H_0^2 (\gamma_2 \cos\omega t + \delta_2 \sin\omega t)[(\underline{H}_0 \times \underline{\Gamma}_2)\cdot \underline{v}_2] \qquad (C.5)$$

and finally

$$\frac{\partial f_2}{\partial t} = \underline{\Gamma_2}\cdot\underline{V_2}\,(-\omega\alpha_2\sin\omega t + \omega\beta_2\cos\omega t) + (\underline{H_0}\times\underline{\Gamma_2})\cdot\underline{V_2}\,(-\omega\xi_2\sin\omega t + \omega\eta_2\cos\omega t) + [\underline{H_0}\times(\underline{H_0}\times\underline{\Gamma_2})]\cdot\underline{V_2}\,[(-\omega r_2\sin\omega t) + \omega\delta_2\cos\omega t] \qquad (C.6)$$

So that to pick the coefficients of Γ_2^2, $\underline{\Gamma_2}\cdot\underline{V_2}$, $(\underline{H_0}\times\underline{\Gamma_2})\cdot\underline{V_2}$ and $[\underline{H_0}\times(\underline{H_0}\times\underline{\Gamma_2})]\cdot\underline{V_2}$ one has to look in (C.4), (C.5), and (C.6). In this fashion the equations obtained are:

$$\underline{\Gamma_2}\cdot\underline{V_2}\left\{\cos\omega t\,\frac{\partial f_2^{(0)}}{\partial v_2} + \omega\Gamma_2(-\alpha_2\sin\omega t + \beta_2\cos\omega t)\right\} = \iiint[f_1'(\alpha_2\cos\omega t + \beta_2\sin\omega t)(\underline{\Gamma_2}\cdot\underline{V_2}') - f_1(\alpha_2\cos\omega t + \beta_2\sin\omega t)(\underline{\Gamma_2}\cdot\underline{V_2})]\,g\,b\,db\,d\epsilon\,d\underline{V_1} \qquad (C.7)$$

$$(\underline{H_0}\times\underline{\Gamma_2})\cdot\underline{V_2}\left[\frac{e_2}{m_2}(\alpha_2\cos\omega t + \beta_2\sin\omega t) - \frac{e_2}{m_2}H_0^2(\eta_2\cos\omega t + \delta_2\sin\omega t) + \omega(-\xi_2\sin\omega t + \eta_2\cos\omega t)\right] = \iiint\{f_1'(\xi_2\cos\omega t + \eta_2\sin\omega t)[(\underline{H_0}\times\underline{\Gamma_2})\cdot\underline{V_2}'] - f_1(\xi_2\cos\omega t + \eta_2\sin\omega t)[(\underline{H_0}\times\underline{\Gamma_2})\cdot\underline{V_2}]\}\,g\,b\,db\,d\epsilon\,d\underline{V_1} \qquad (C.8)$$

$$[\underline{H_0}\times(\underline{H_0}\times\underline{\Gamma_2})]\cdot\underline{V_2}\left\{\frac{e_2}{m_2}(\xi_2\cos\omega t + \eta_2\sin\omega t) + \omega(-r_2\sin\omega t + \delta_2\cos\omega t)\right\} = \iiint\{f_1'(r_2\cos\omega t + \delta_2\sin\omega t)[(\underline{H_0}\times(\underline{H_0}\times\underline{\Gamma_2}))\cdot\underline{V_2}'] - f_1(r_2\cos\omega t + \delta_2\sin\omega t)[(\underline{H_0}\times(\underline{H_0}\times\underline{\Gamma_2}))\cdot\underline{V_2}]\}\,g\,b\,db\,d\epsilon\,d\underline{V_1} \qquad (C.9)$$

and

$$\Gamma_2^2\cos\omega t\left[(\alpha_2\cos\omega t + \beta_2\sin\omega t) + \frac{v_2}{3}\left(\frac{\partial\alpha_2}{\partial v_2}\cos\omega t + \frac{\partial\beta_2}{\partial v_2}\sin\omega t\right) - H_0^2\sin^2\psi\left\{(r_2\cos\omega t + \delta_2\sin\omega t) + \frac{v_2}{3}\left(\frac{\partial r_2}{\partial v_2}\cos\omega t + \frac{\partial\delta_2}{\partial v_2}\sin\omega t\right)\right\}\right] = \iiint(f_1'f_2'^{(0)} - f_1f_2^{(0)})\,g\,b\,db\,d\epsilon\,d\underline{V_1} \qquad (C.10)$$

The formidable integrals on the right have been evaluated by Chapman and Cowling (Ref:s) using several approximations. First of all, $\underline{V}_2 = \underline{V}_2' = g$ (the relative velocity). This is equivalent to the assumption of infinite mass for the heavy neutral particles (anologous to Rutherford scattering). Secondly, f, is assumed to be Maxwellian as was pointed out in Ch. II, Section A. This considerably simplifies the integrals; for example, the one in (C.7) reduces to

$$(\alpha_2 \cos\omega t + \beta_2 \sin\omega t)(\underline{V}_2' \cdot \underline{V}_2) \iiint (f_1' - f_1) g \, b \, db \, d\epsilon \, d\underline{V}_1 \qquad \text{(C.11)}$$

The last integral is equal to $\frac{1}{v_2} \frac{\partial f_2^{(0)}}{\partial v_2} = -\frac{v_2}{\lambda}$ according to Chapman and Cowling. (C.8) and (C.9) can be treated the same way. In equation (C.10) a different procedure is followed: each side is multiplied by $d\underline{V}_2 = 4\pi v_2^2 \, dv_2$; the resulting integral on the right has also been evaluated by Chapman and Cowling and is

$$\iiint (f_1' f_2'^{(0)} - f_1 f_2^{(0)}) g \, b \, db \, d\epsilon \, d\underline{V}_1 \, d\underline{V}_2 = \left(\frac{kT}{m_1 \lambda} v_2^3 \frac{\partial f_2^{(0)}}{\partial v_2} + \frac{m_2 v_2^4}{m_1 \lambda} f_2^{(0)}\right)(4\pi) \qquad \text{(C.12)}$$

All these equations lead to a system of linear equations for $\alpha_2, \beta_2, \xi_2, \eta_2, \tau_2, \delta_2$. The system is:

$$\frac{\partial f_2^{(0)}}{\partial v_2} + \omega v_2 \beta_2 = -\frac{\alpha_2 v_2^2}{\lambda} \qquad \text{(C.13)}$$

$$\omega v_2 \alpha_2 = \frac{\beta_2 v_2^2}{\lambda} \qquad \text{(C.14)}$$

$$\frac{\alpha_2 e_2}{m_2} - \frac{e_2}{m_2} H_0^2 \tau_2 + \omega \eta_2 = -\frac{v_2 \xi_2}{\lambda} \qquad \text{(C.15)}$$

$$\frac{e_2}{m_2}(\beta_2 - H_0^2\delta_2) - \omega\xi_2 = -\frac{v_2\eta_2}{\lambda} \tag{C.16}$$

$$\frac{e_2}{m_2}\xi_2 + \omega\delta_2 = -\frac{v_2\gamma_2}{\lambda} \tag{C.17}$$

$$\frac{e_2}{m_2}\eta_2 - \omega\gamma_2 = -\frac{v_2\delta_2}{\lambda} \tag{C.18}$$

Solution of this system gives the equations presented in Chapter II.

APPENDIX D

The Sen-Wyller Formula

In Chapter II the dispersion relation for the index of refraction of the medium was given as:

$$\begin{vmatrix} -(\epsilon_I + \epsilon_{III}\cos^2\phi) & \epsilon_{II}\cos\phi & \epsilon_{III}\sin\phi\cos\phi \\ -\epsilon_{II}\cos\phi & M^2-(\epsilon_I+\epsilon_{III}) & \epsilon_{II}\sin\phi \\ \epsilon_{III}\sin\phi\cos\phi & -\epsilon_{II}\sin\phi & M^2-(\epsilon_I+\epsilon_{III}\sin^2\phi) \end{vmatrix} = 0 \qquad (2.57)$$

Evaluation of the determinant leads to:

$$-(\epsilon_I+\epsilon_{III}\cos\phi)\left\{[M^2-(\epsilon_I+\epsilon_{III})][M^2-(\epsilon_I+\epsilon_{III}\sin^2\phi)]+\epsilon_{II}^2\sin^2\phi\right\}+$$

$$\epsilon_{II}\cos\phi\left\{[\epsilon_{II}\cos\phi][M^2-(\epsilon_I+\epsilon_{III}\sin^2\phi)]+\epsilon_{II}\epsilon_{III}\sin^2\phi\cos\phi\right\}+$$

$$\epsilon_{III}\sin\phi\cos\phi\left\{\epsilon_{II}^2\sin\phi\cos\phi-\epsilon_{III}\sin\phi\cos\phi[M^2-(\epsilon_I+\epsilon_{III})]\right\}=0$$

Algebraic Manipulation of the previous equation leads to:

$$M^4\left(-\frac{D}{2}-\frac{E}{2}\cos^2\phi\right)+M^2\left(A+B\cos^2\phi\right)-\frac{D}{2}\left[B+\frac{A}{2}\right]=0$$

Where A, B, C, D, E are as defined in Chapter II. If the quadratic equation for M^2 is solved by the usual formula, then

$$M^2=\frac{-(A+B\cos^2\phi)\pm\sqrt{A^2+2AB\cos^2\phi+B^2\cos^4\phi-D(BD+BE\cos^2\phi+\frac{AD}{2}+\frac{AE\cos^2\phi}{2}}}{-(D+E\cos^2\phi)}$$

If the fact that $2AB-2BD\left(\frac{D+E}{2}\right)=0$ is used along with the equality $C^2=BD-\frac{AED}{2}$ then the quantity under the radical reduces to:

$$B^2\cos^4\phi-C^2\sin^2\phi$$

Therefore

$$M^2 = \frac{A + B\cos^2\phi \mp \sqrt{B^2\cos^4\phi - C^2\sin^2\phi}}{D + E\cos^2\phi}$$

And since the angle of rotation $\phi = \varphi - \pi/2$, where φ is the angle between the propagation vector and the magnetic field, then

$$M^2 = \frac{A + B\sin^2\varphi \mp \sqrt{B^2\sin^4\varphi - C^2\cos^2\theta}}{D + E\sin^2\varphi}$$

This is the Sen-Wyller magneto-ionic formula.

APPENDIX E

A. Dipole Field Deviation

It is desired to prove that for a dipole field the angle between the radial direction and the lines of force is constant.

If a spherical coordinate system is used with origin at the center of the earth then a line of force for a dipole field is given by

$$r = L \sin^2\theta \tag{E.1}$$

Where r is the distance from the center of the earth, θ is the colatitude and L is a parameter which characterizes each line of force.

A function ψ is defined such that

$$\psi = r - L \sin^2\theta \tag{E.2}$$

Since the gradient of a function ψ in a spherical coordinate system is given by

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}, \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\right) \tag{E.3}$$

Then

$$\nabla\psi = (1, -2\cot\theta, 0) = \left(1, -\frac{2L\sin\theta\cos\theta}{r}, 0\right) \tag{E.4}$$

or using (E.1)

$$\nabla\psi = (1, -2\cot\theta, 0)$$

The vector $\nabla\psi$ is normal to ψ = constant, in particular, it is normal to $\psi = 0$, that is, normal to the lines of force.

For spherical surfaces, i.e. $\varphi = r$

$$\nabla\varphi = (1, 0, 0) \tag{E.5}$$

Hence $\nabla\psi \cdot \nabla r = 1$ = a constant. Any vector $\underline{\ell}$ parallel to the lines of force at a point satisfies the condition $\underline{\ell} \cdot \nabla\psi = 0$.

In particular, $\underline{\ell}$ can be the magnetic field, since

$$\underline{H} \cdot \nabla\psi = \left(\frac{-2M\cos\theta}{r^3}, \frac{-M\sin\theta}{r^3}, 0\right) \cdot (1, -2\cot\theta, 0) \tag{E.6}$$

$$\underline{H} \cdot \nabla\psi = -\frac{2M\cos\theta}{r^3} + \frac{2M\cos\theta}{r^3} \tag{E.7}$$

$$\underline{H} \cdot \nabla\psi = 0$$

Therefore, $\underline{H} \times (\nabla r \times \nabla\psi) = (\underline{H} \cdot \nabla\psi)\nabla r - (\underline{H} \cdot \nabla r)\nabla\psi$ (E.8)

or by (E.7)

$$\underline{H} \times (\nabla r \times \nabla\psi) = -(\underline{H} \cdot \nabla r)\nabla\psi \tag{E.9}$$

but $\underline{H} \cdot \nabla r = H_r$ and $\nabla r \times \nabla\psi = (0, 0, -2\cot\theta)$.

Therefore

$$(H_r, H_\theta, 0) \times (0, 0, 2\cot\theta) = H_r \nabla\psi$$

$$(2\cot\theta H_\theta, 2H_r\cot\theta, 0) = H_r \nabla\psi \tag{E.10}$$

Two equations result

$$2\cot\theta H_\theta = +H_r \tag{E.11}$$

$$H_r = H_r \tag{E.12}$$

The second one is trivial, but from the first one, one can see that since $H_r = H\cos\xi$ where ξ is the angle that $\underline{H}$ makes with the radial direction then

$$\cos\xi = \frac{2\cot\theta H_\theta}{H} \tag{E.13}$$

or

$$\cos\xi = \frac{-2\cos\theta}{(1+3\cos^2\theta)^{1/2}} \qquad \text{(E.14)}$$

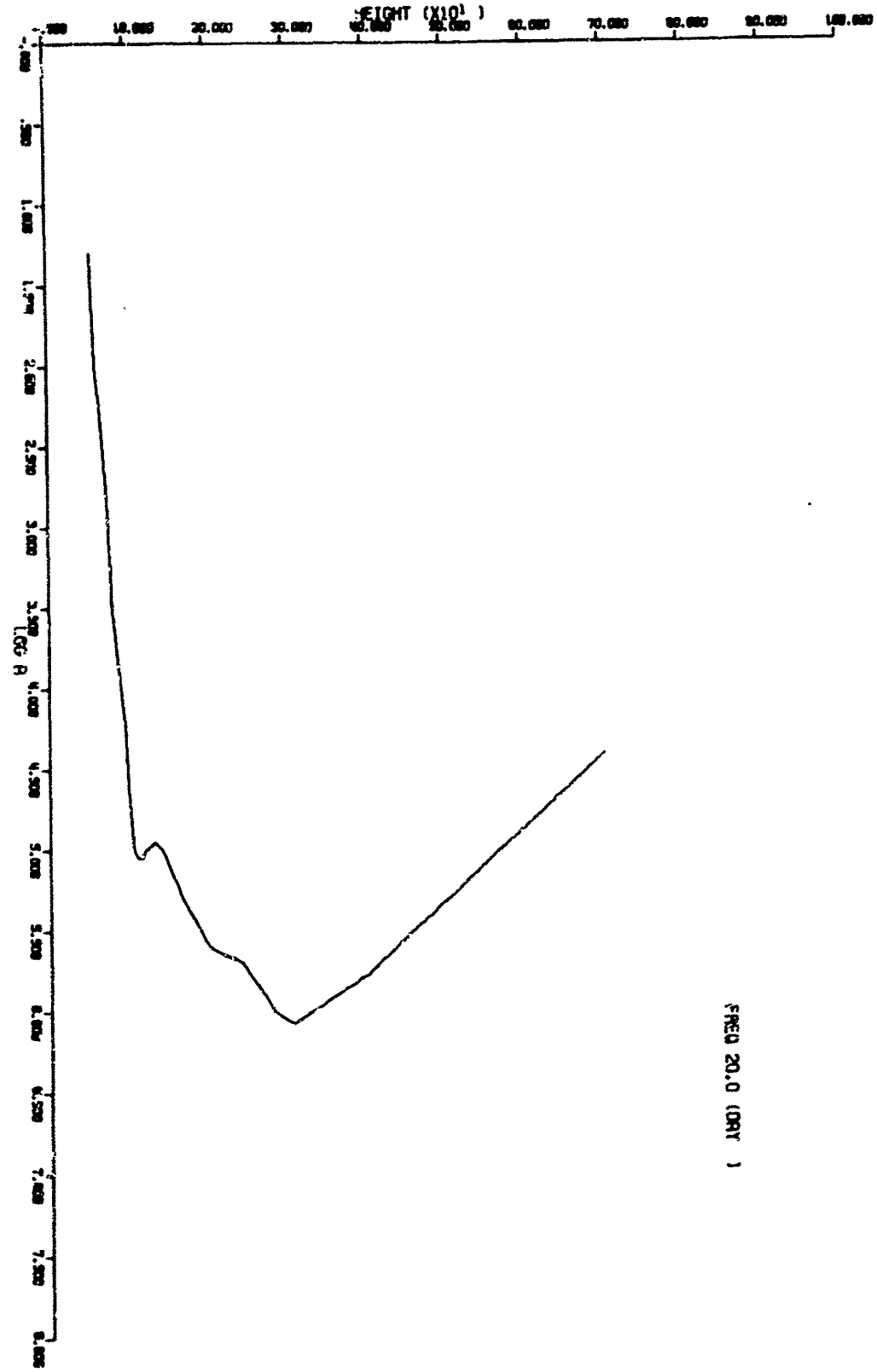
from which it is evident that ξ is independent of ν which was to be proved.

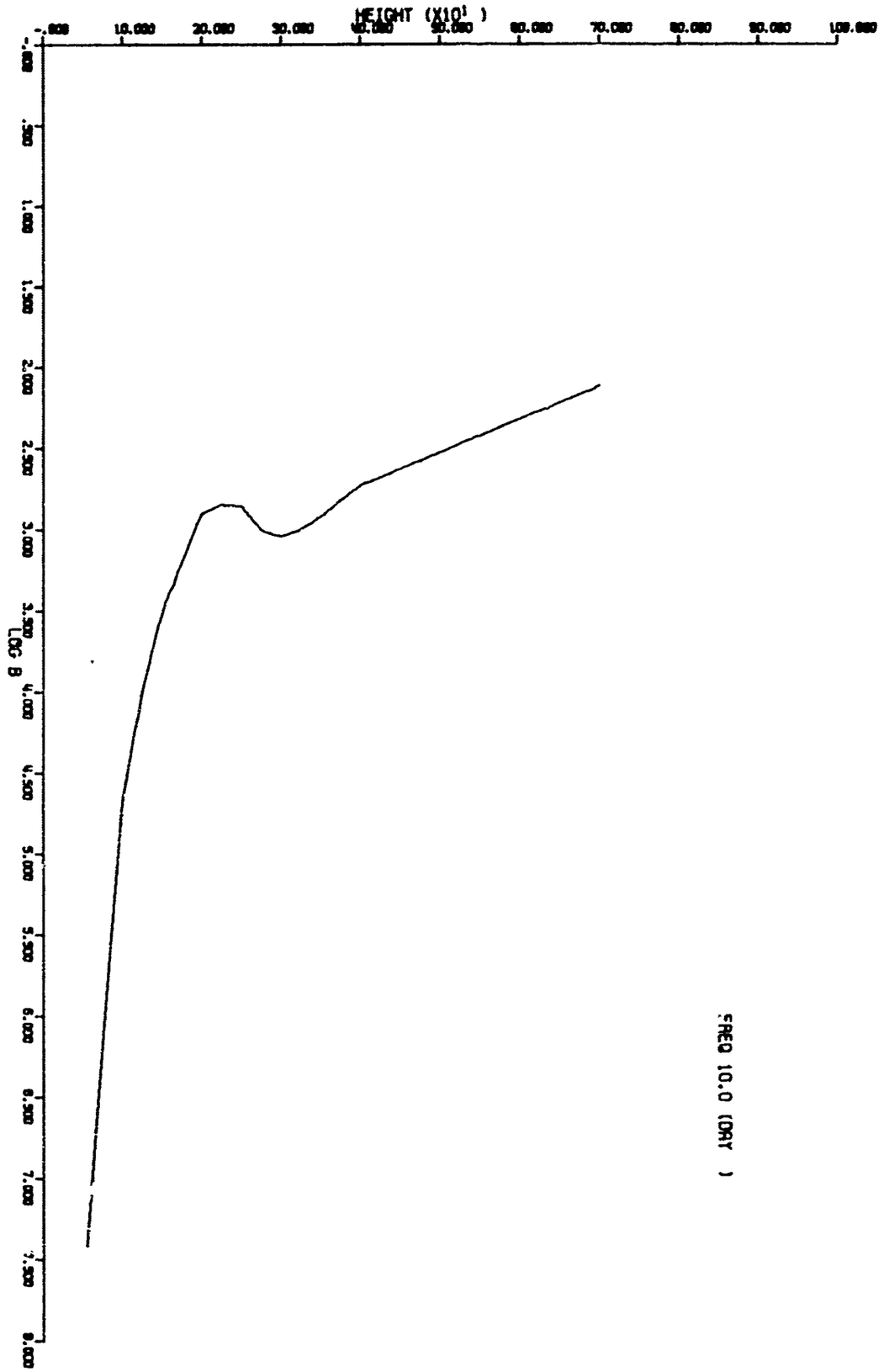
APPENDIX F

Individual Height Profiles of μ, χ

Because they are so numerous, it was decided to place the results in an Appendix. The figures are almost self-explanatory once it is known what the symbols stand for. A stands for electron density, B for collision frequency, U, X for μ and γ respectively while + and - denote ordinary and extraordinary rays respectively. The frequencies and the conditions (day-time or night-time) for which the profile is computed are set down at the top of the graph. The ordinate in all of these is the height. The graphs are arranged in blocks, one for each abscissa, and written each in a progression of frequencies from the lower to the higher values.

SP/PH/65-1

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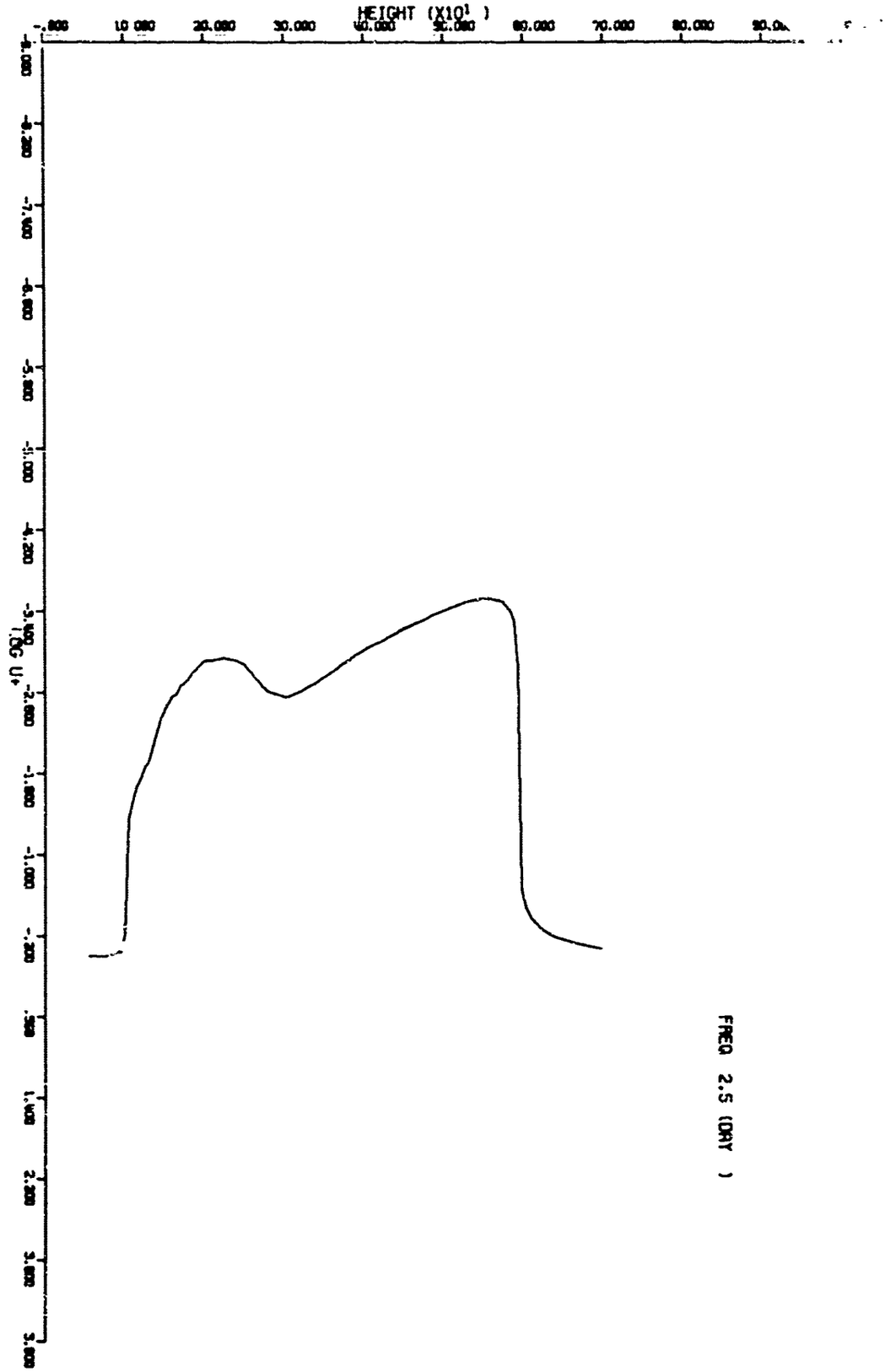
30.000
70.000
80.000
90.000
LOG B
FREQ 10.0 (DAY)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20.000
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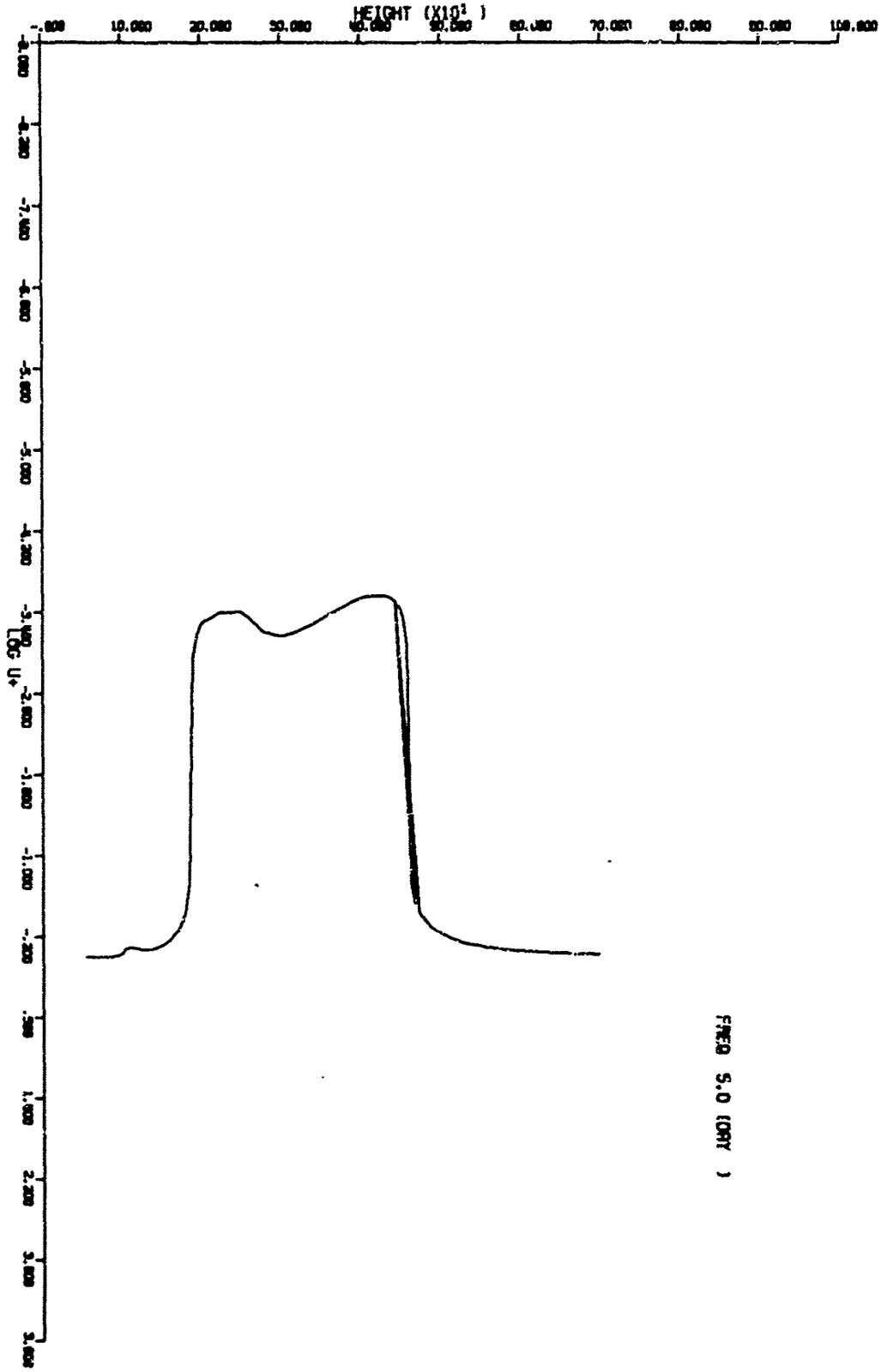
50.000
60.000
70.000
80.000
90.0
LOG U
FREQ 2.5 (DAY)

SP/PH/65-1



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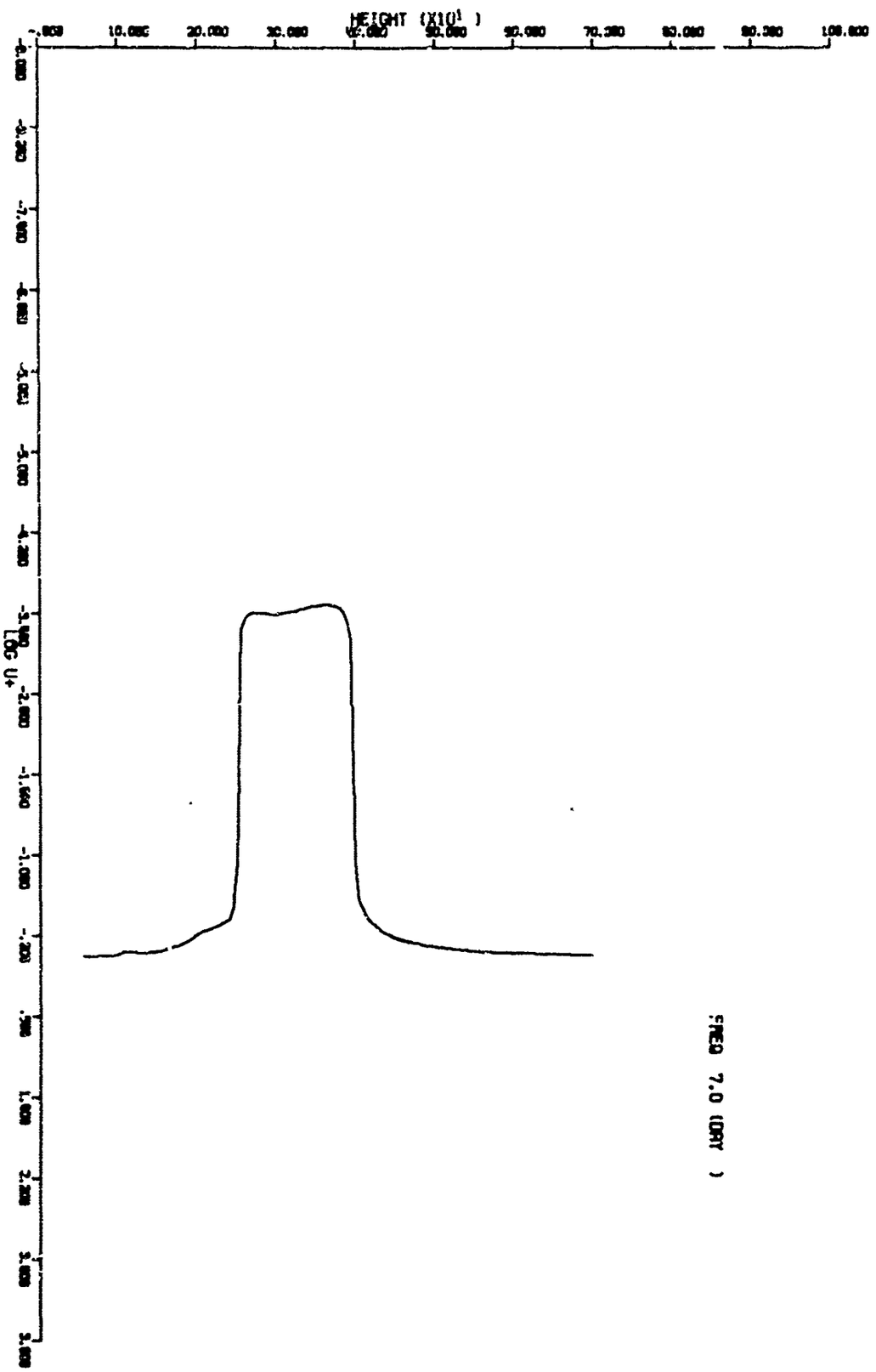
HEIGHT (X10¹)
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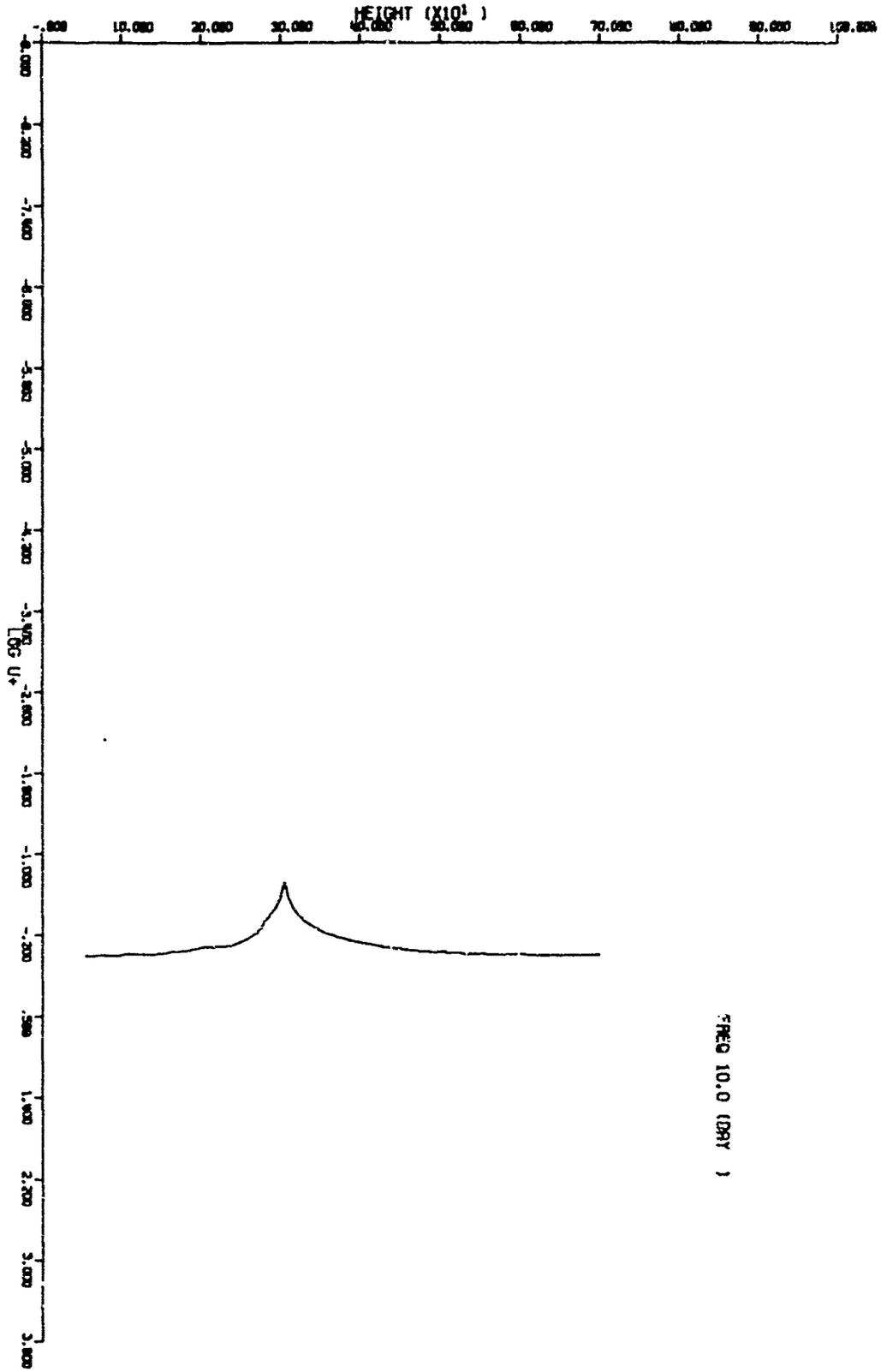
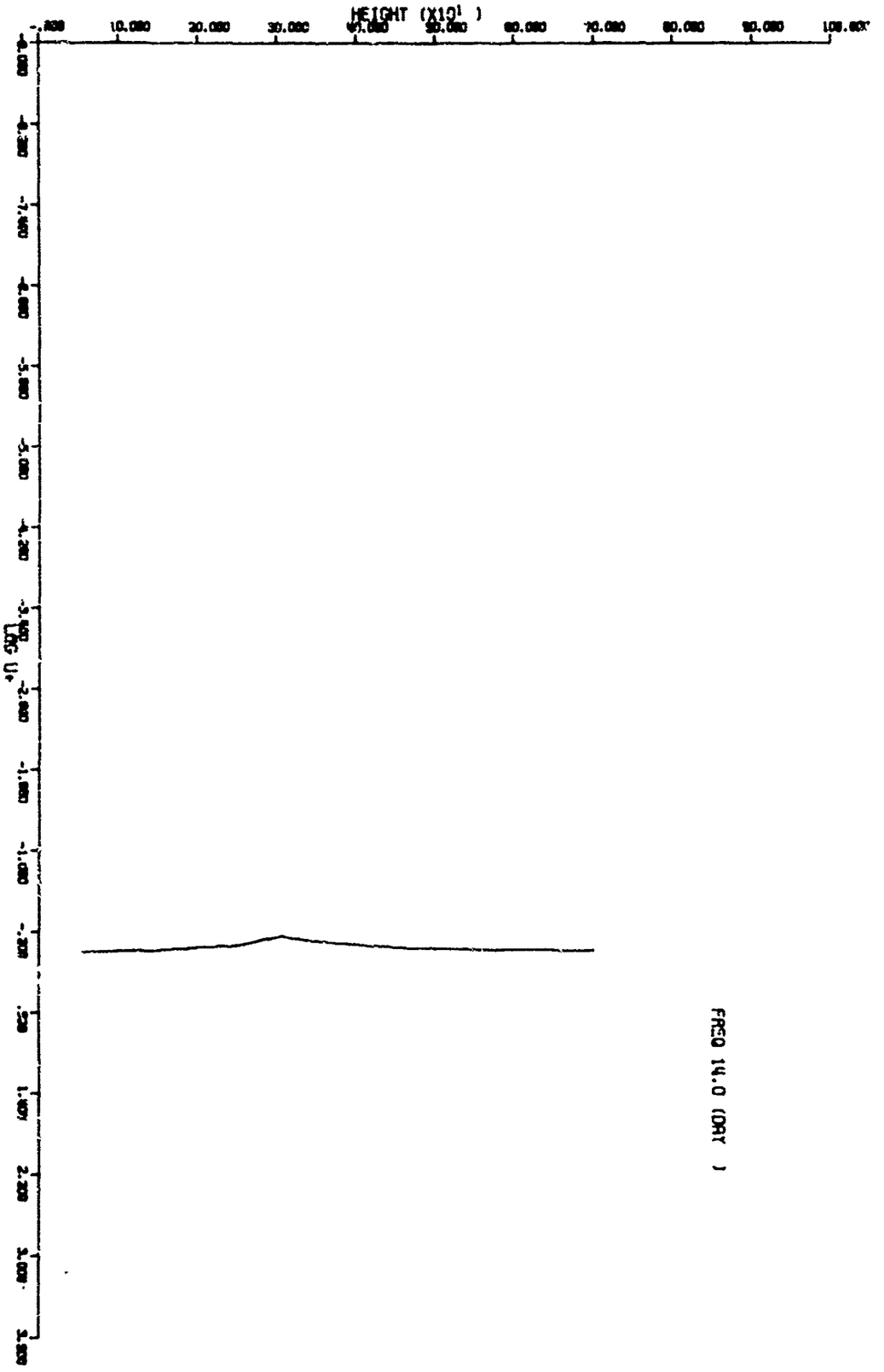
20.000
30.000
70.000
80.000
100.000
LOG U+
-2.000
-1.000
1.000
2.000
3.000
FED 7.0 (DRY)

SP/PH/65-1

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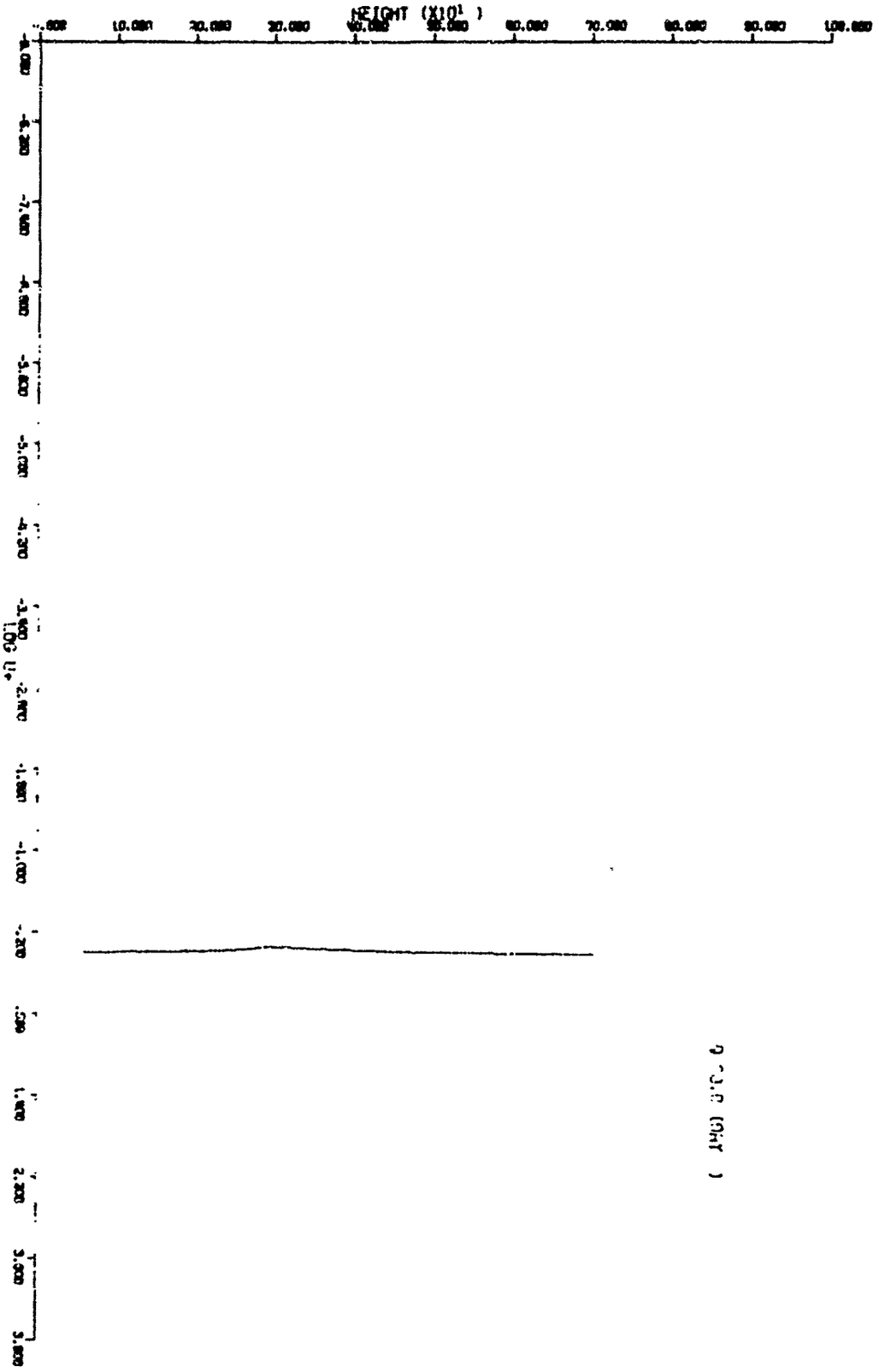
30.000
70.000
80.000
90.000
FREQ 14.0 (DRY)


HEIGHT (X10¹)
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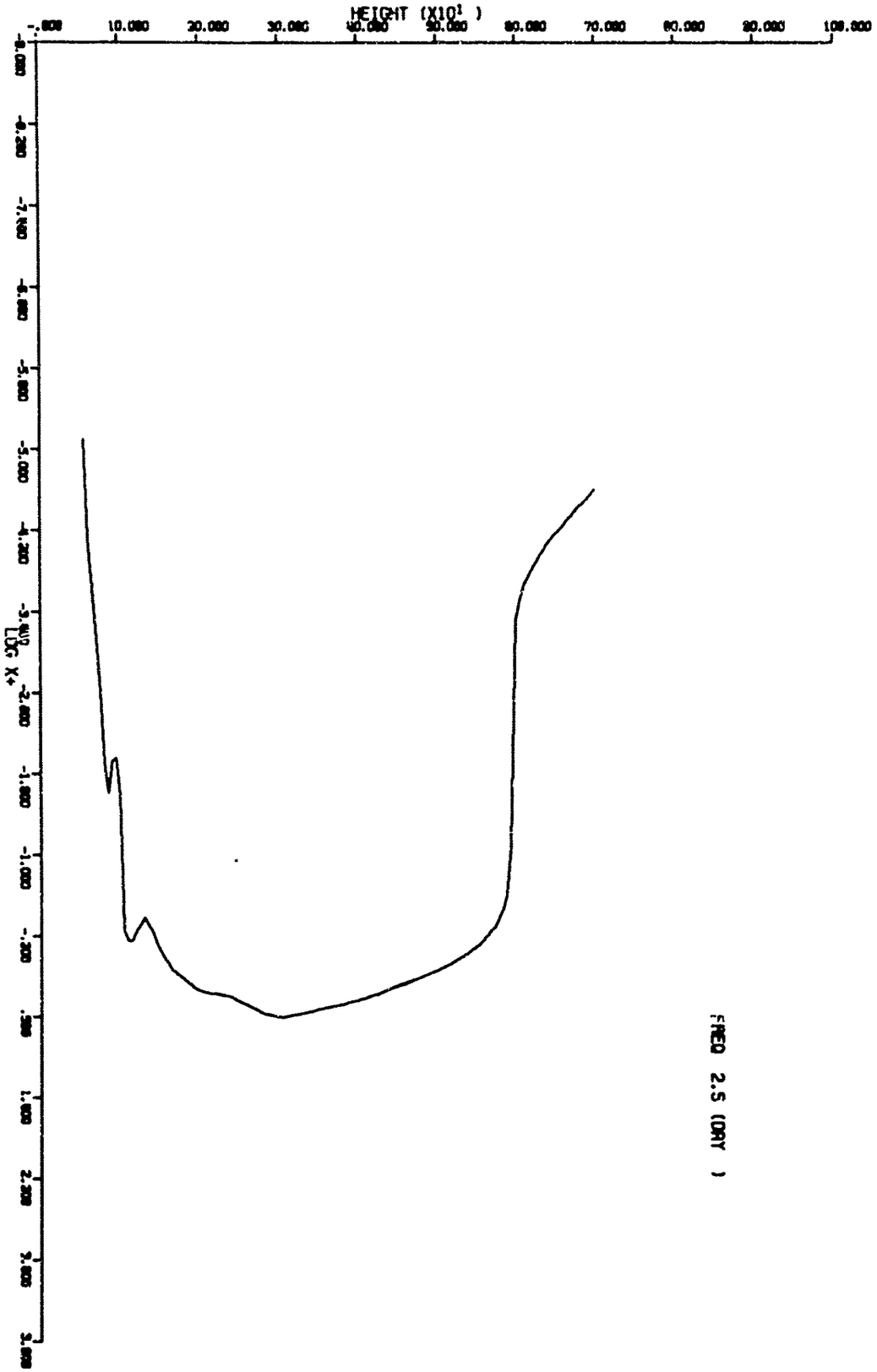
80.000
90.000
100.000
LOG (H)

SP/PH/65-1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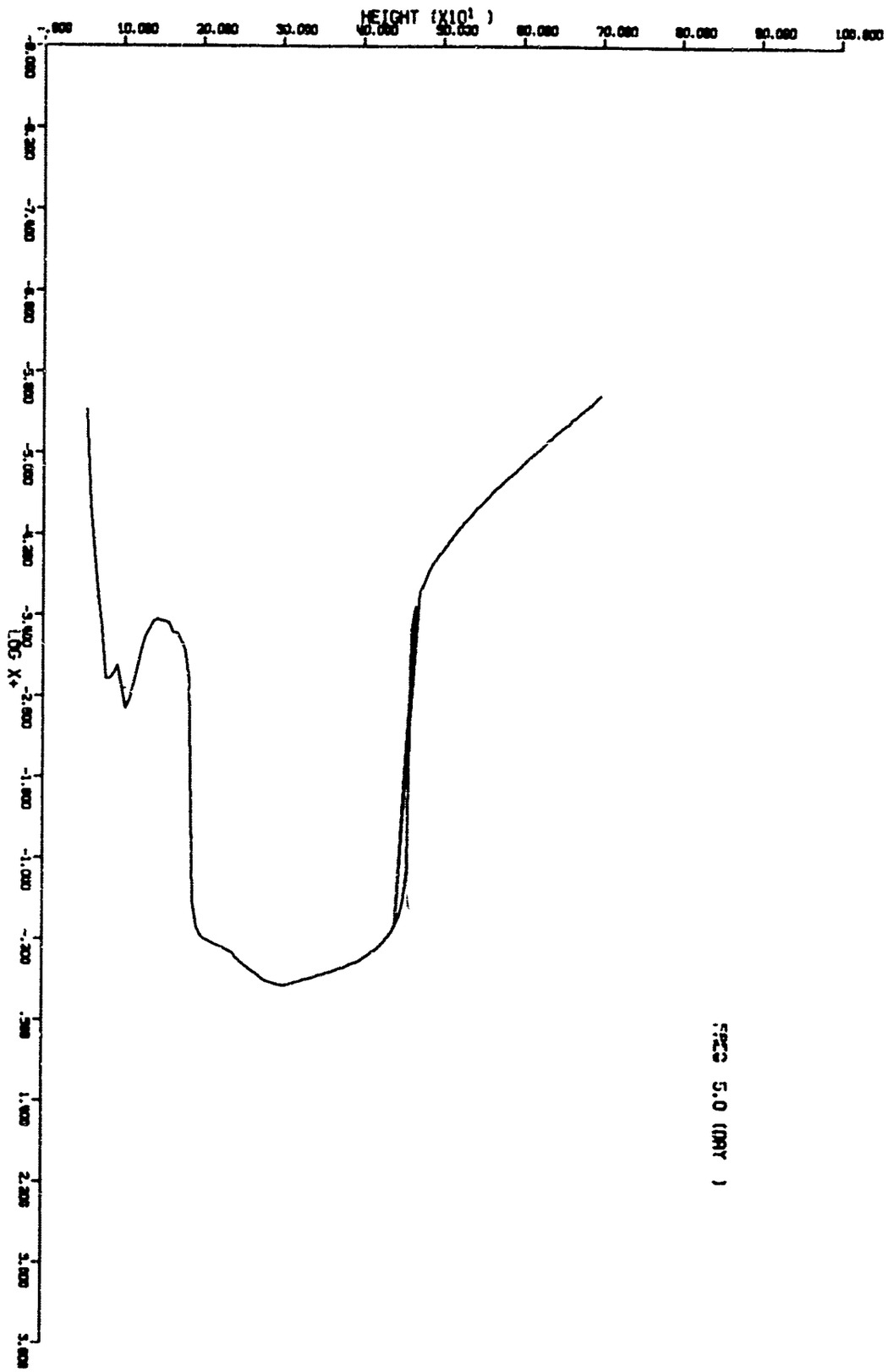
20.000
30.000
70.000
80.000
90.000
100.000
LOG X+
FREQ 2.5 (DAY)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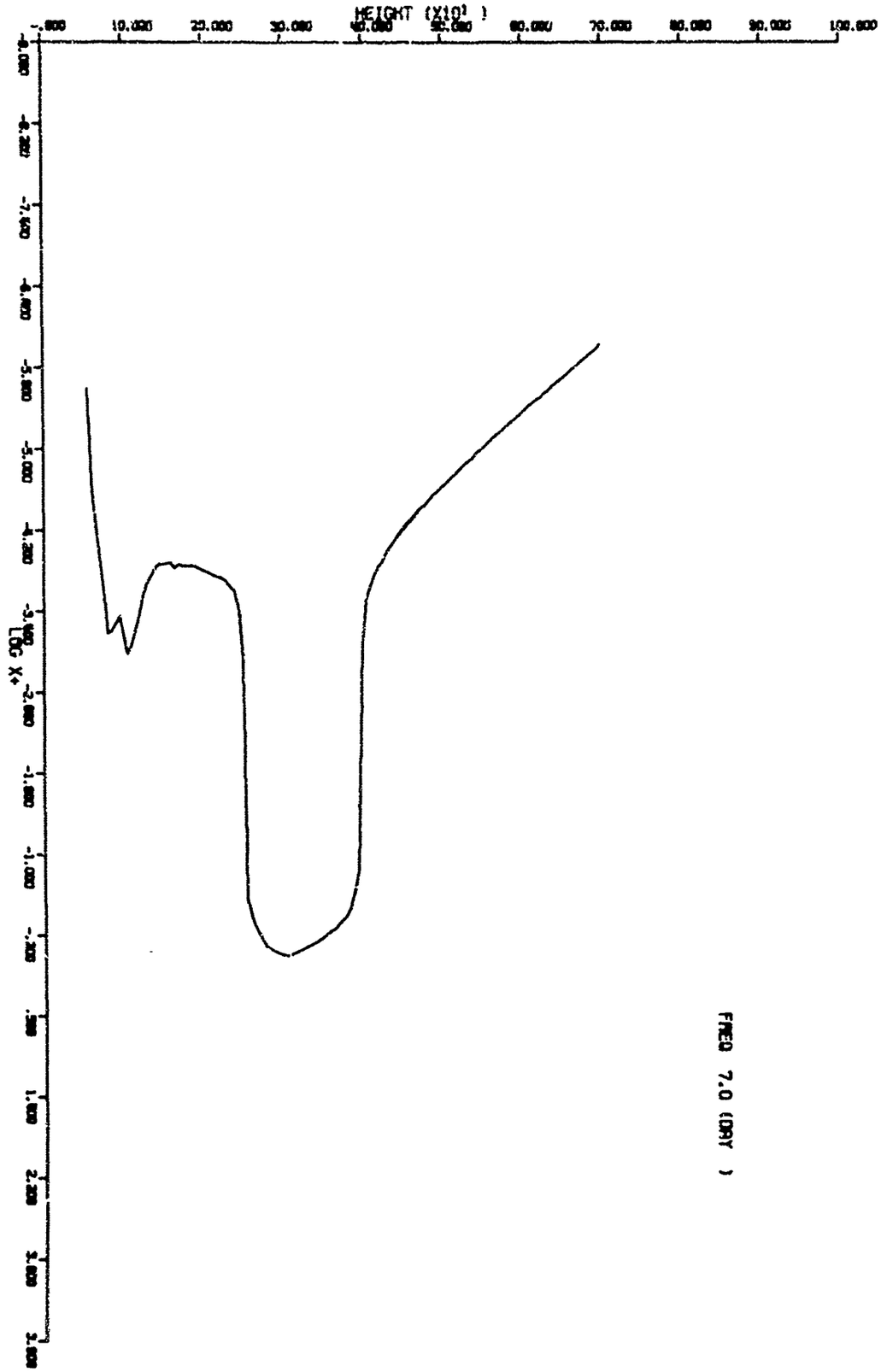
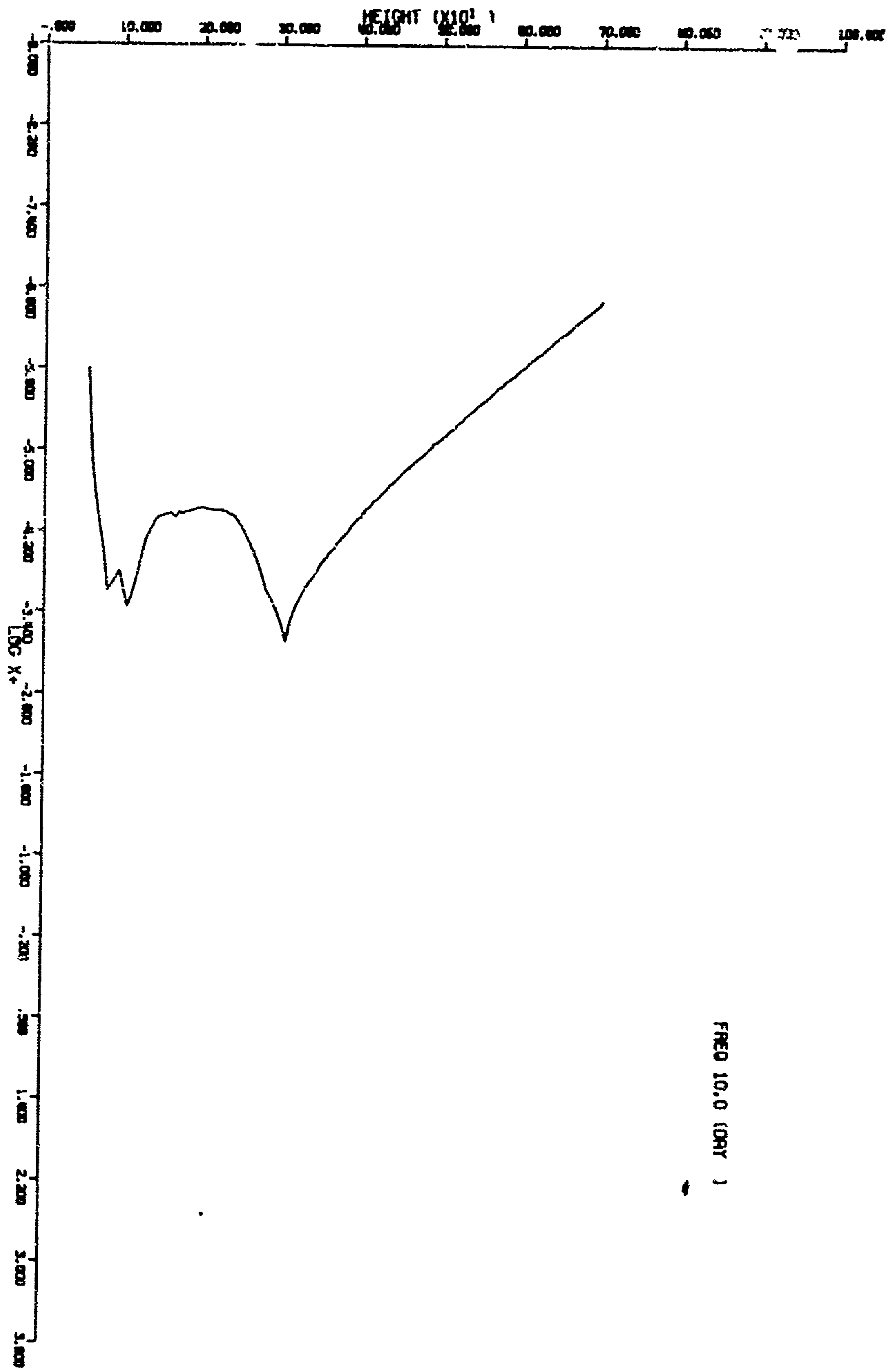
80.000
90.000
100.000
LOG X+
-2.000
-1.000
1.000
2.000
3.000

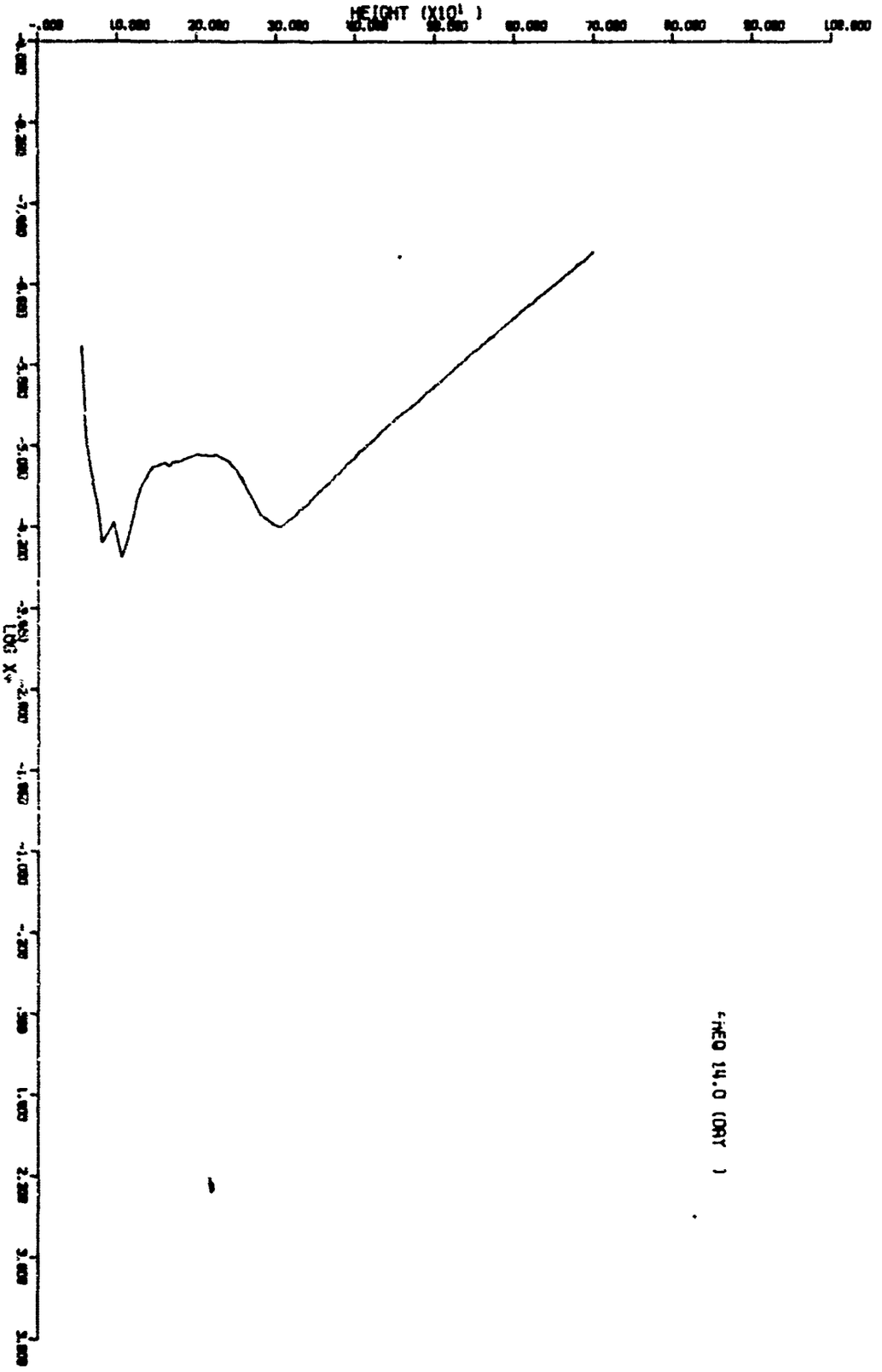
5.0 (DAY)

SP/PH/65-1



FREQ 7.0 (DAY)


HEIGHT (X10¹)
LOG X+
FREQ 10.0 (DRY)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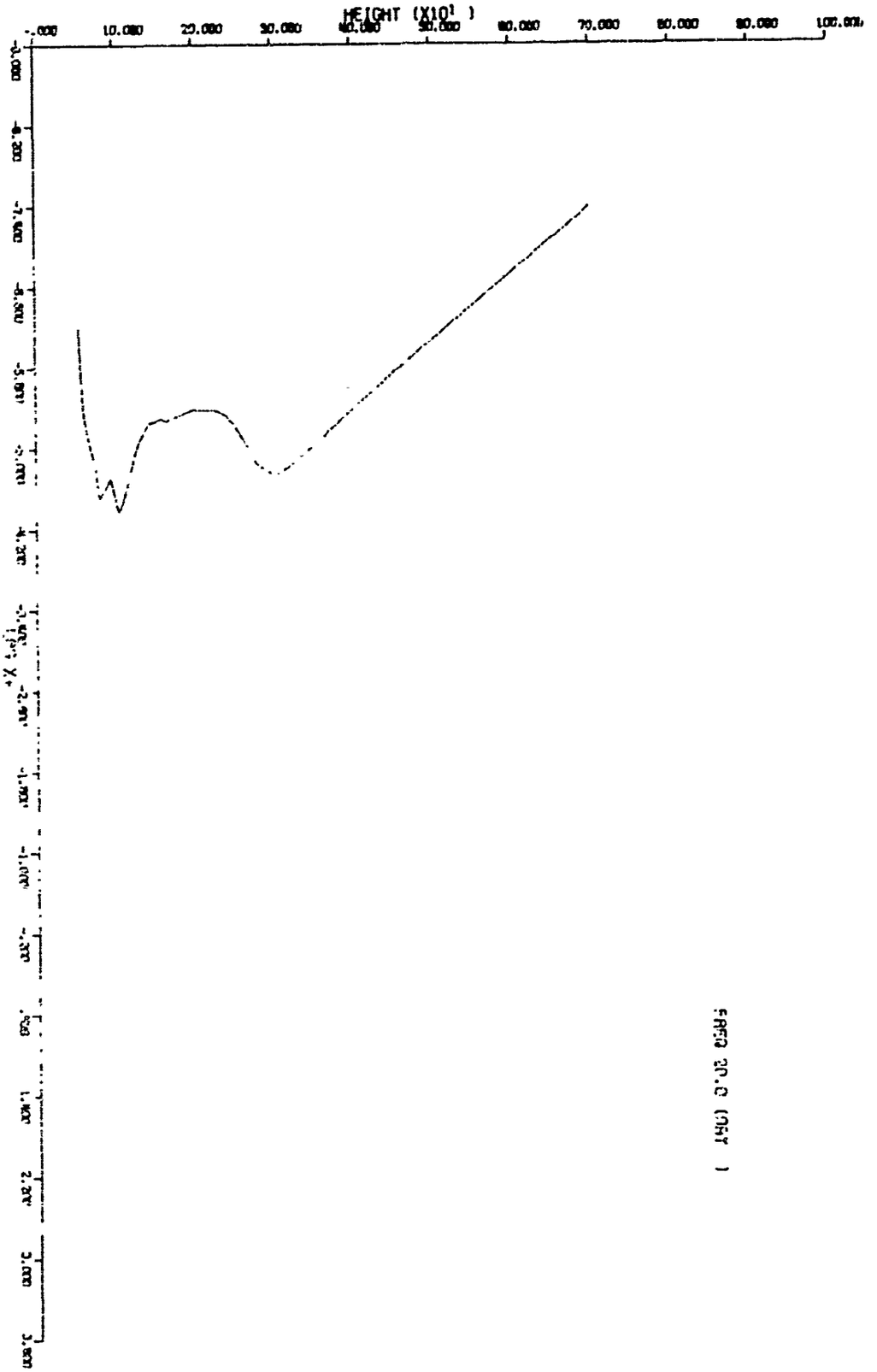
90.000
100.000
[CO] X 10⁶
THEO 14.0 (DRY)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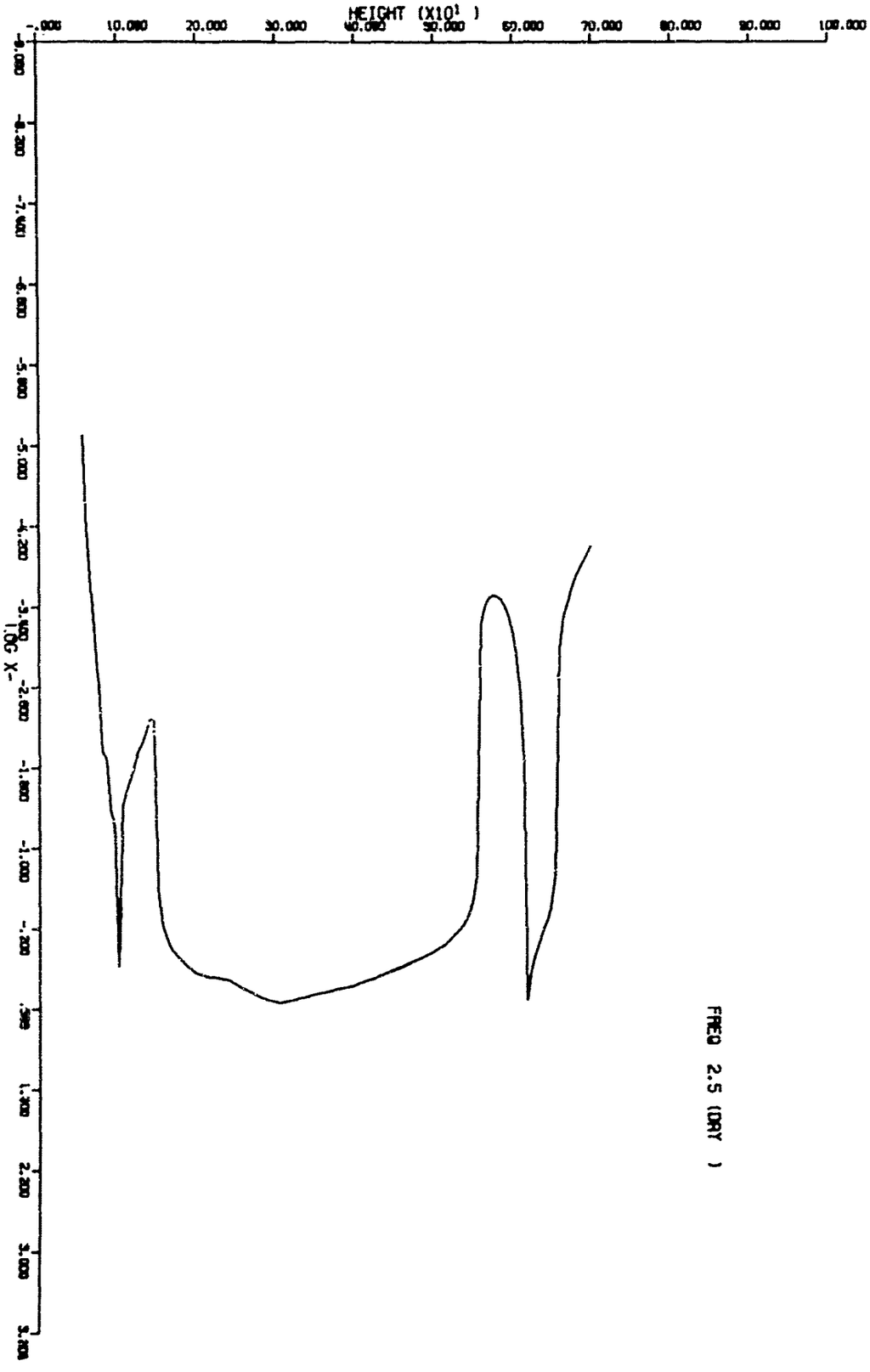
100.000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-3.400
-2.600
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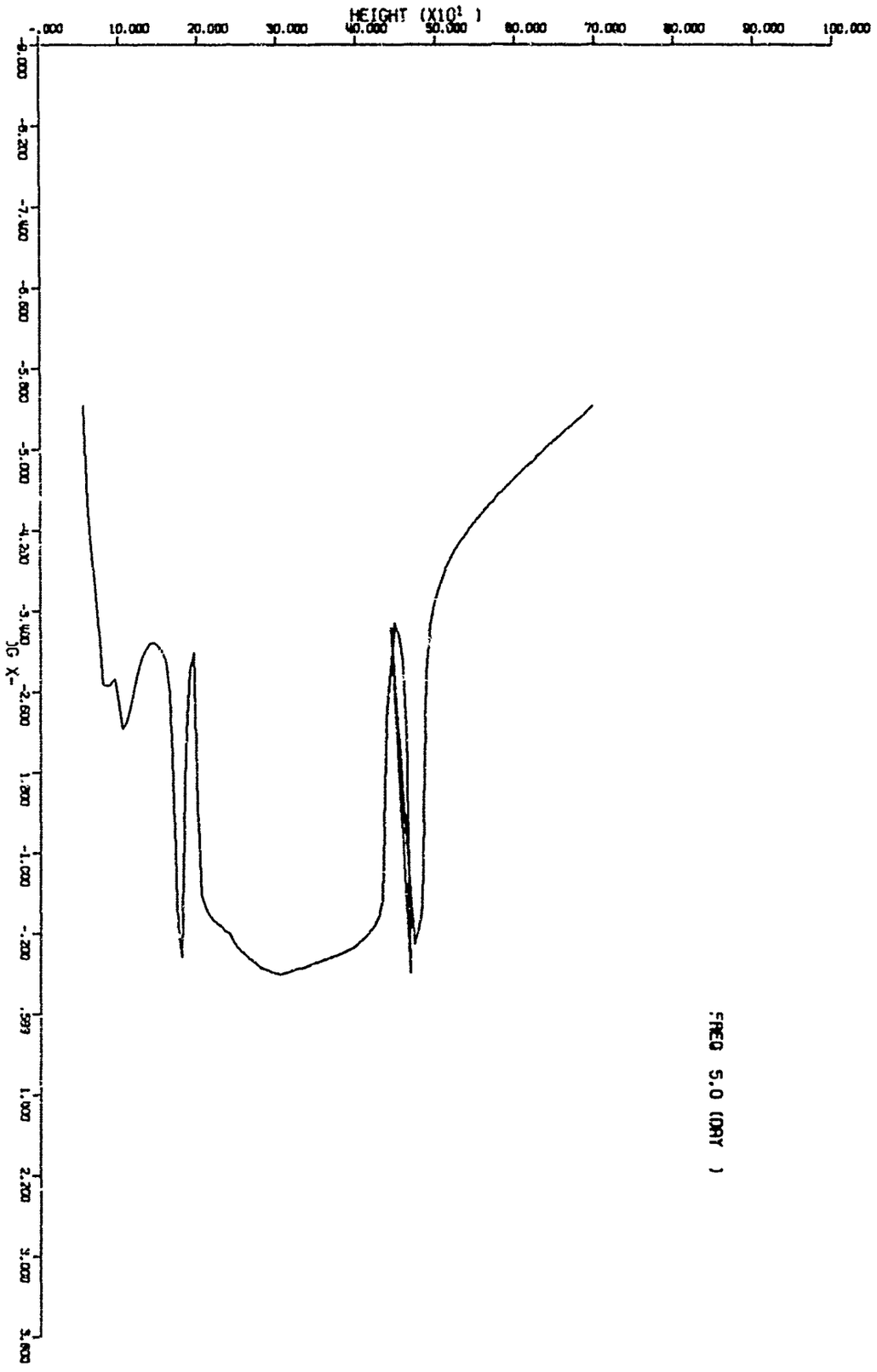
3.000
3.800
1.0G X
FREQ 2.5 (DRY)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-3.400
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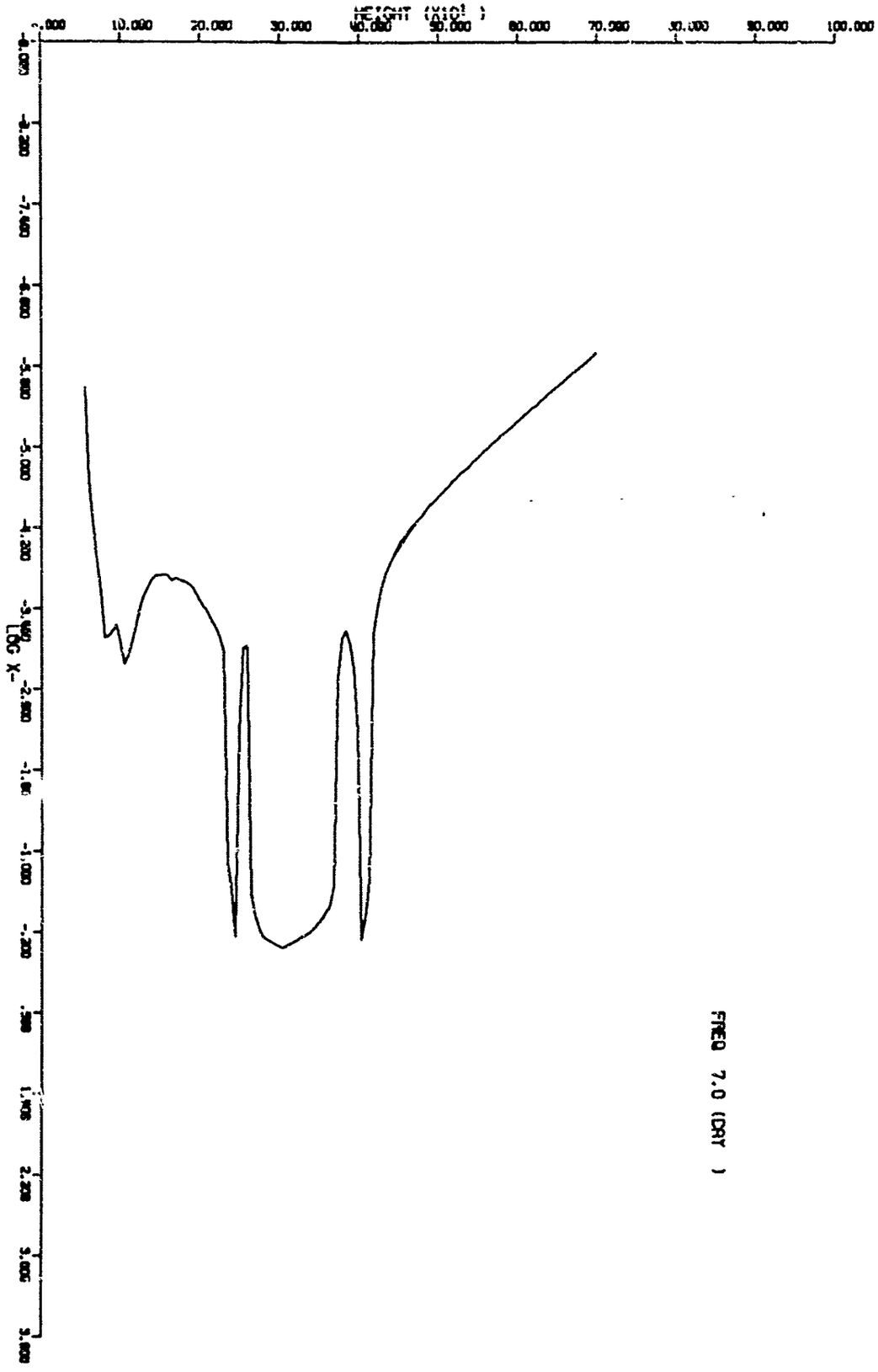
3.000
3.800
JG X-
FREQ 5.0 (DAY)


.000
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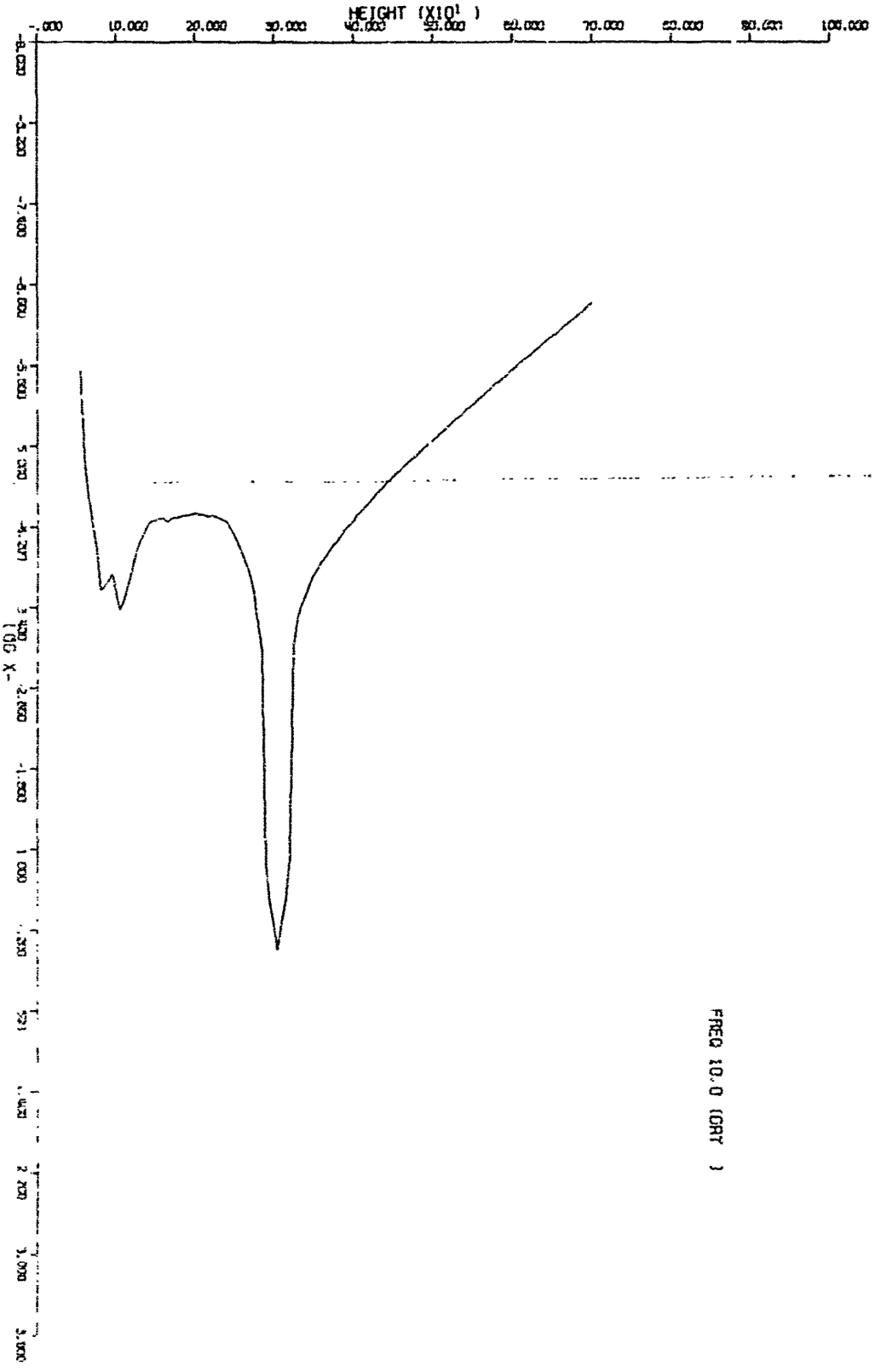
100.000
LOG X-
FREQ 7.0 (DRY)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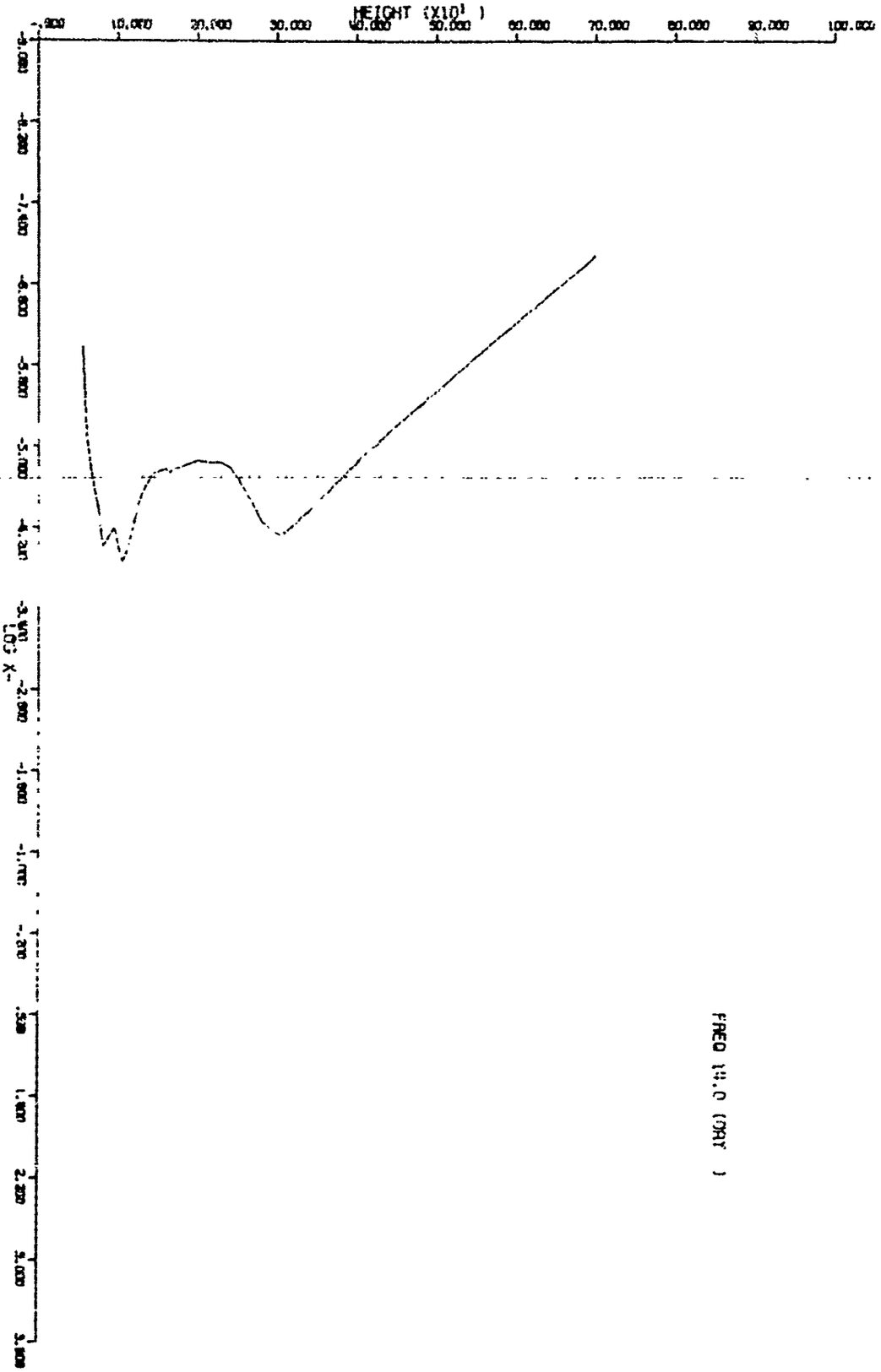
3.000
dB X-
FREQ 10.0 (GHz)

SP/PH/6 -1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-5.000
-4.000
-3.000
-2.000
-1.000
-.200
.500
1.000
2.000
3.000
3.500
LOG X-
FREQ (1/DAY)

HEIGHT (X10¹)
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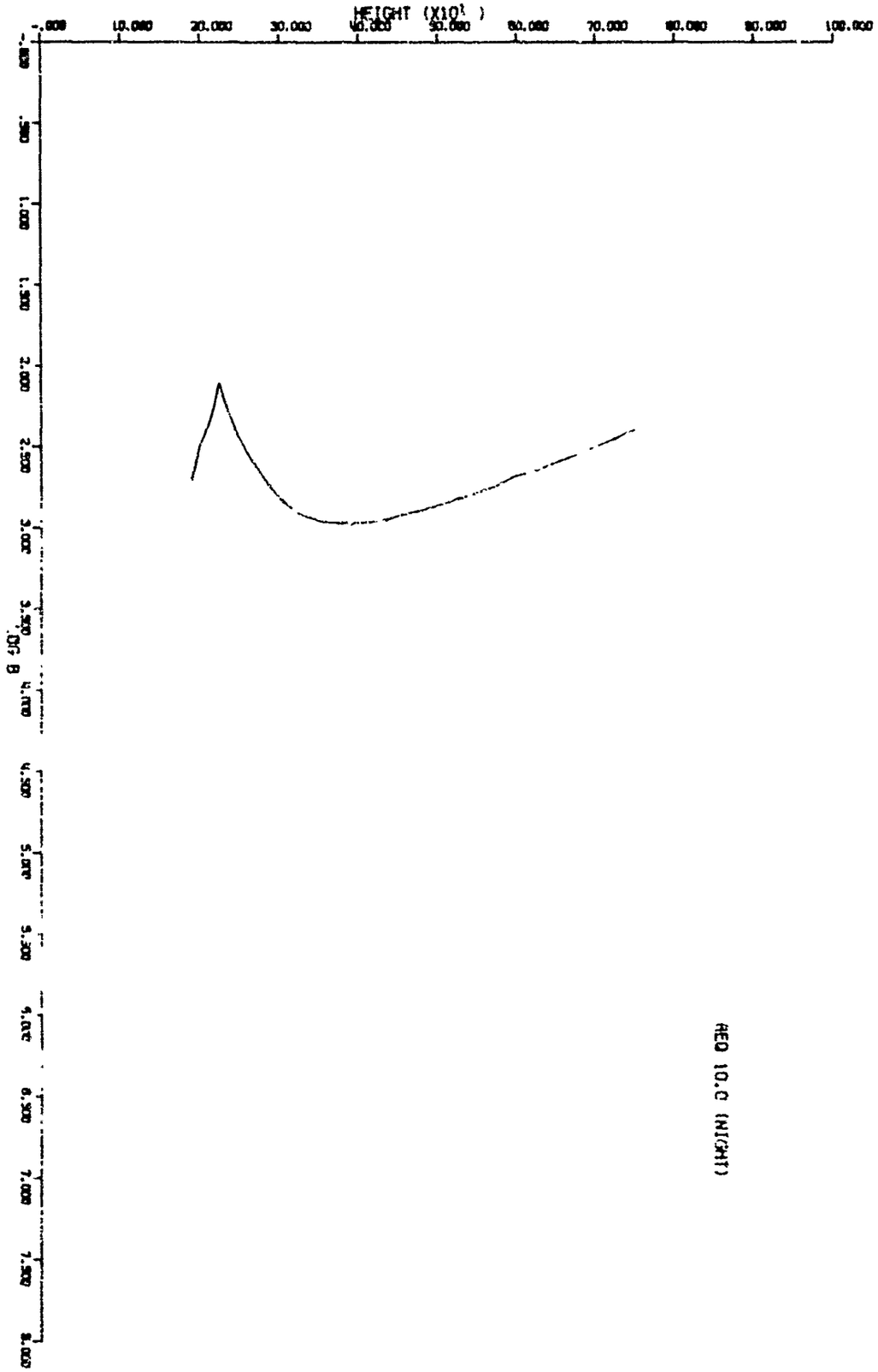
20.000
50.000
70.000
LOG X-
AREA 10.0 (DAY)

SP/PH/65-1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80.000
90.000
100.000
REQ 10.0 (NIGHT)

HEIGHT (X10³)
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20.000
30.000
70.000
80.000
100.000
R (CM)
FREQ 14.0 (NIGHT)

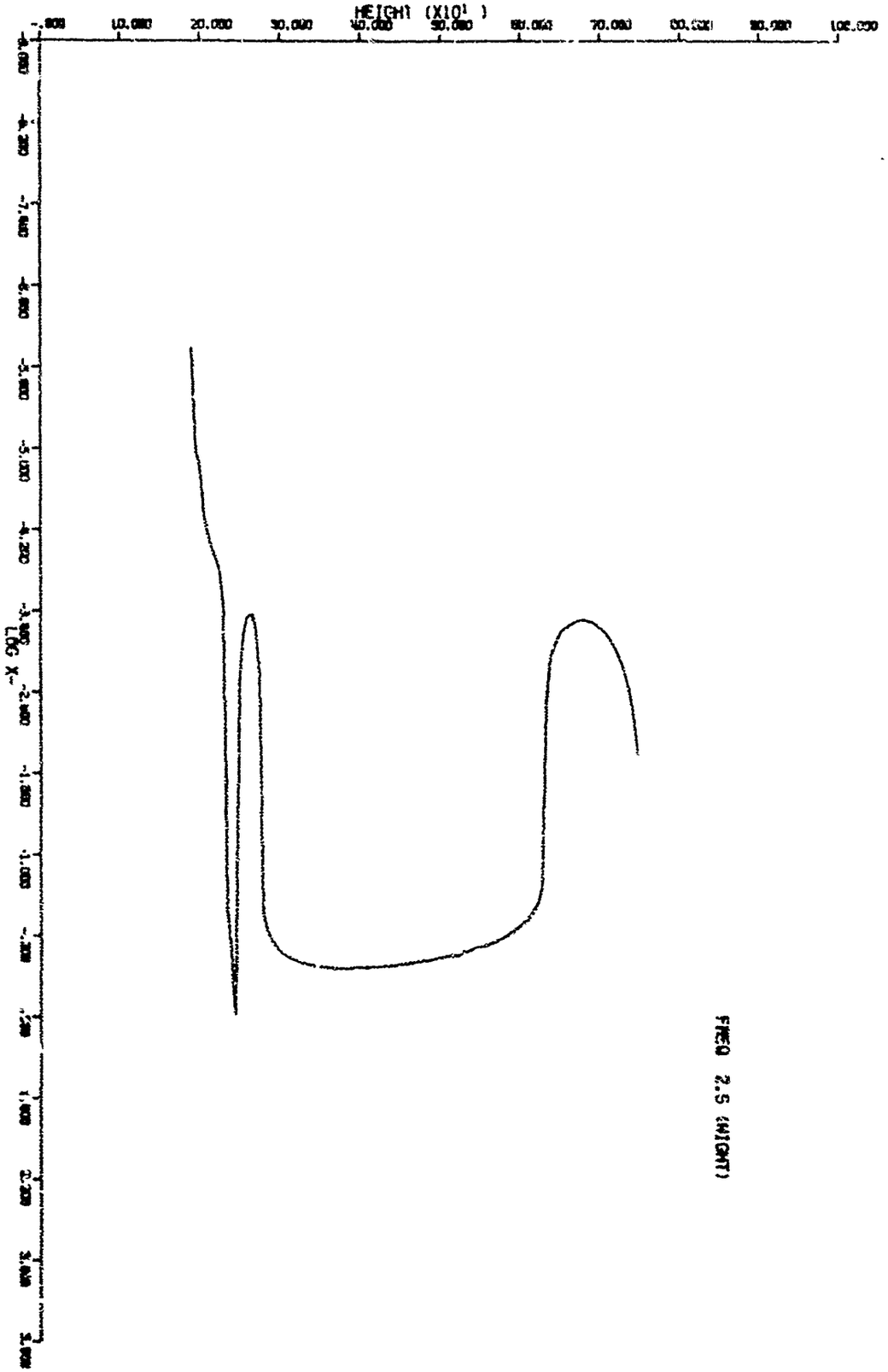
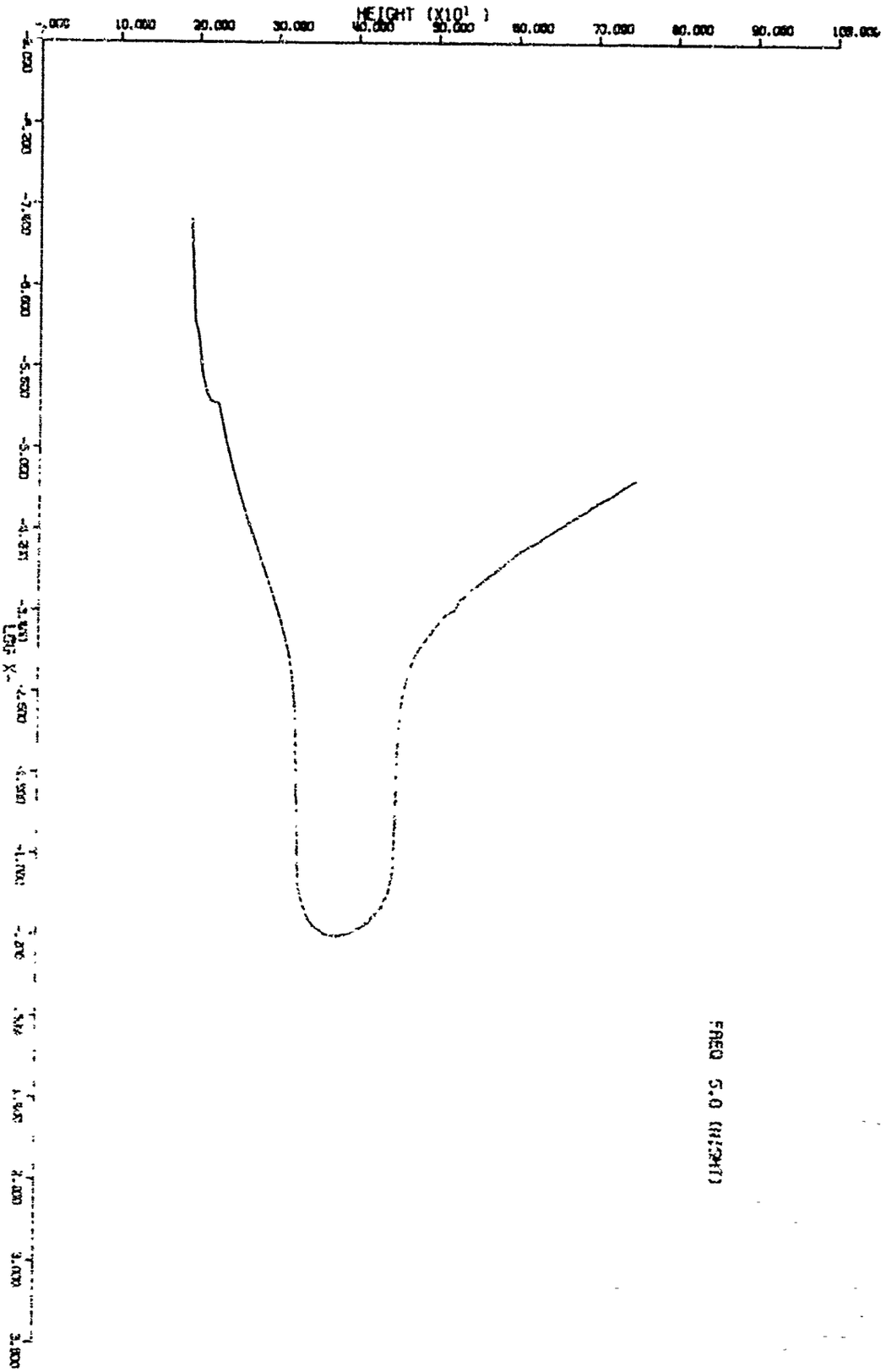
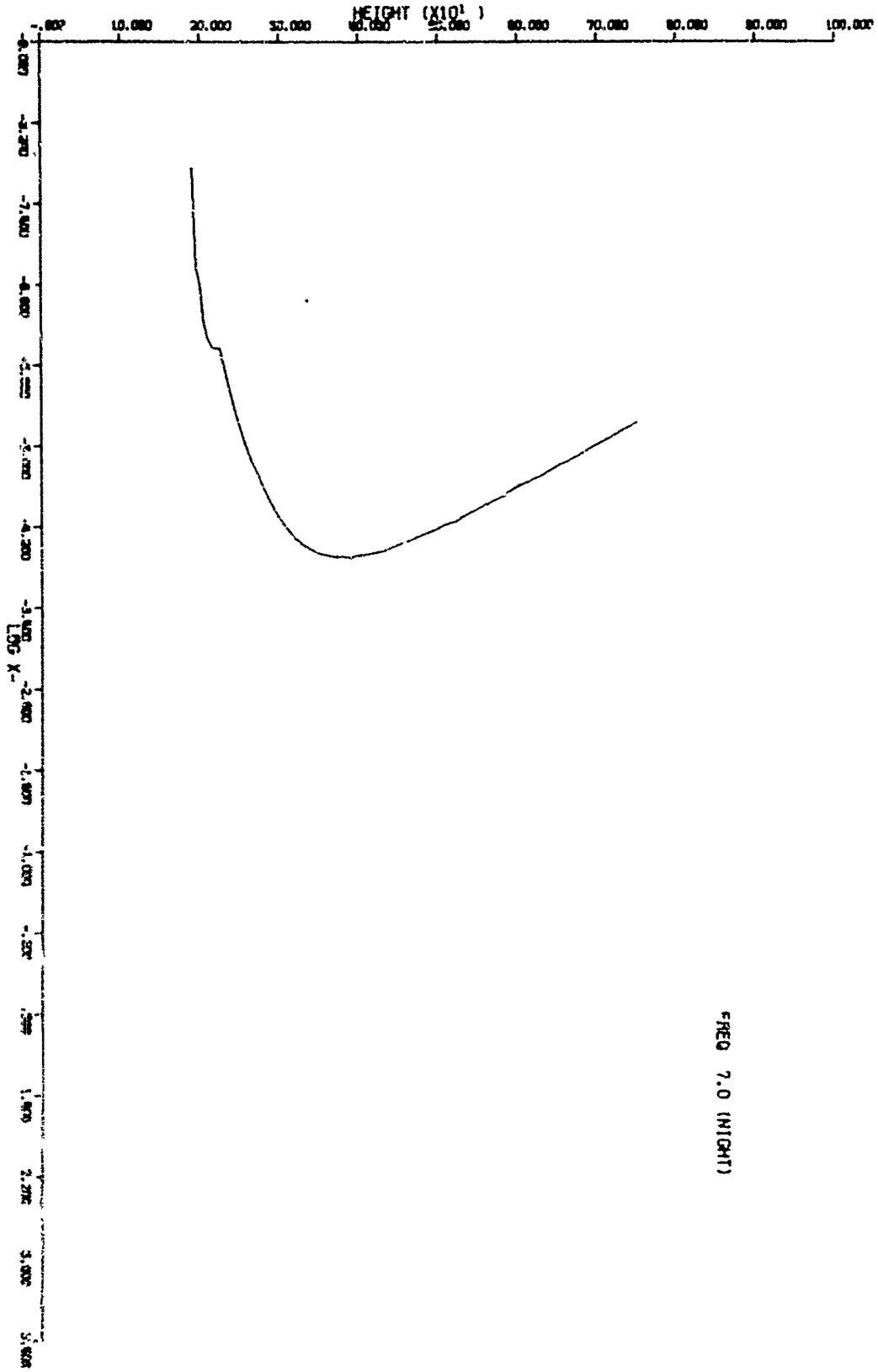
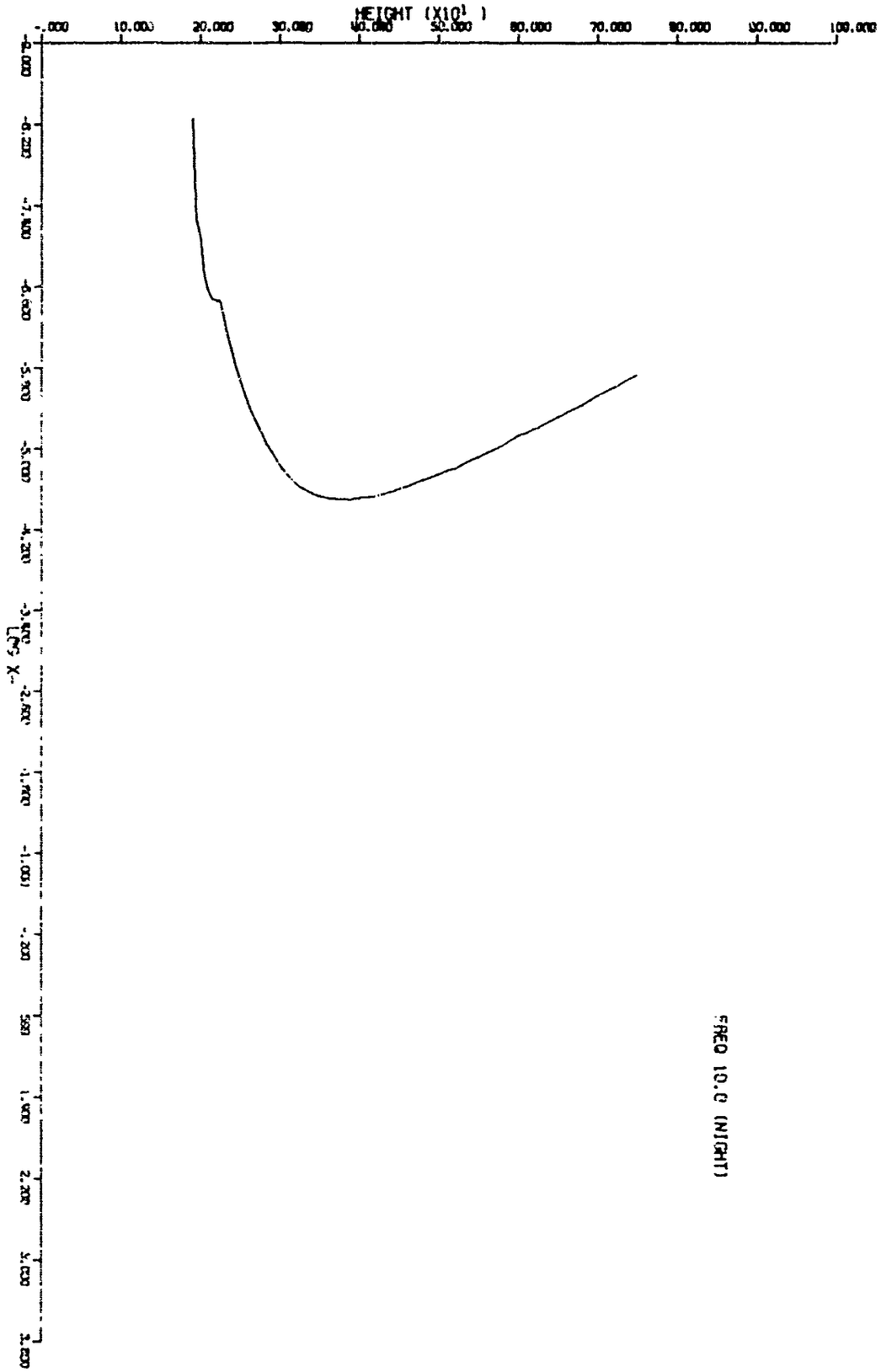


FIGURE 2.5 (WEIGHT)

JP/PH/65-1


HEIGHT (X10¹)
10.000
20.000
30.000
40.000
50.000
60.000
70.000
80.000
90.000
100.000
FREQ 5.0 (KHZ)


HEIGHT (X10¹)
10.000
20.000
30.000
40.000
50.000
60.000
70.000
80.000
90.000
100.000
LOG X-
FREQ 7.0 (NIGHT)


HEIGHT (X10¹)
-.000
10.000
20.000
30.000
40.000
50.000
60.000
70.000
80.000
90.000
100.000
-8.000
-7.500
-7.000
-6.500
-6.000
-5.500
-5.000
-4.500
-4.000
-3.500
-3.000
-2.500
-2.000
-1.500
-1.000
-.500
.000
.500
1.000
1.500
2.000
2.500
3.000
LOG X-
FREQ 10.0 (NIGHT)

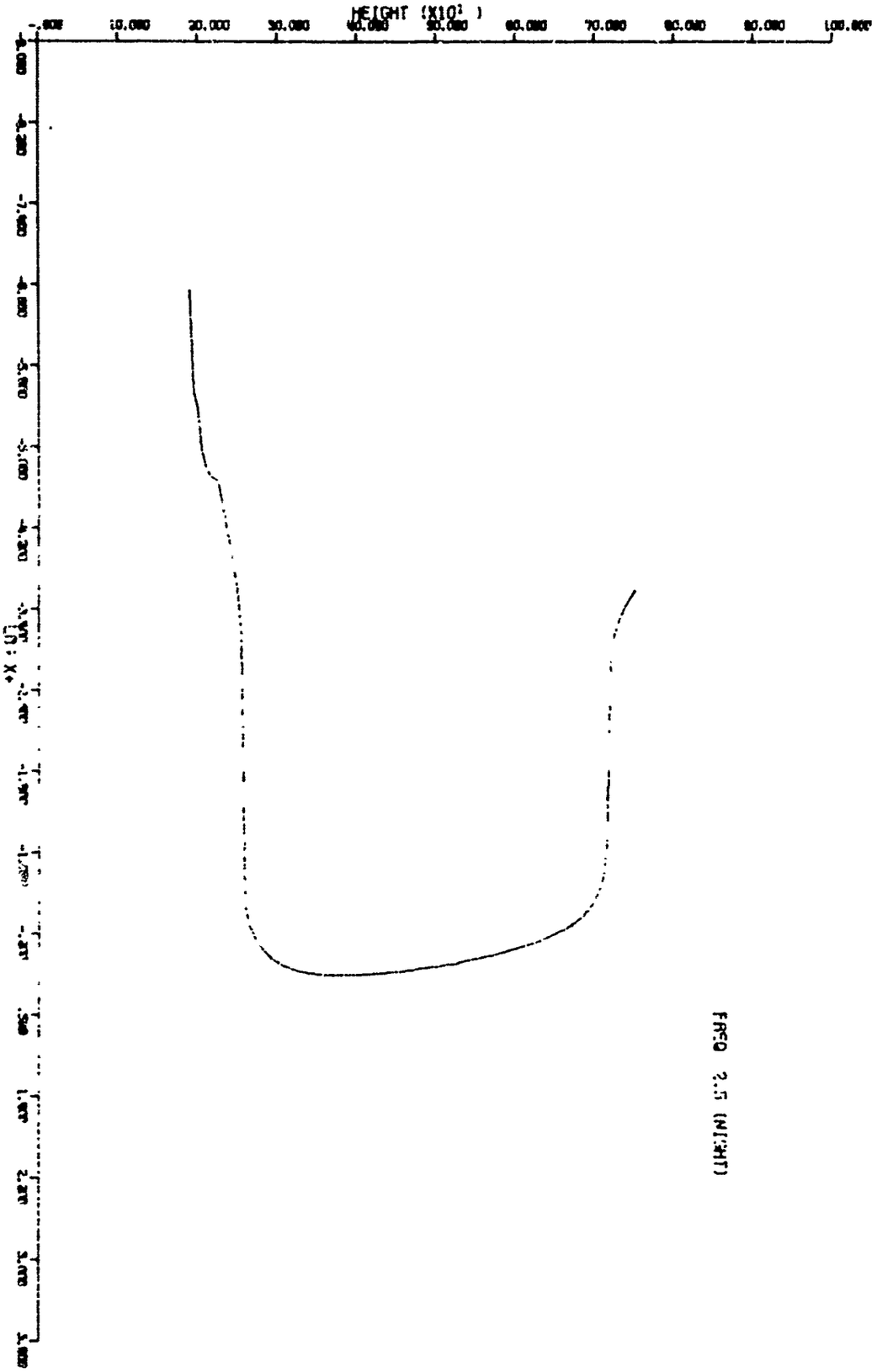
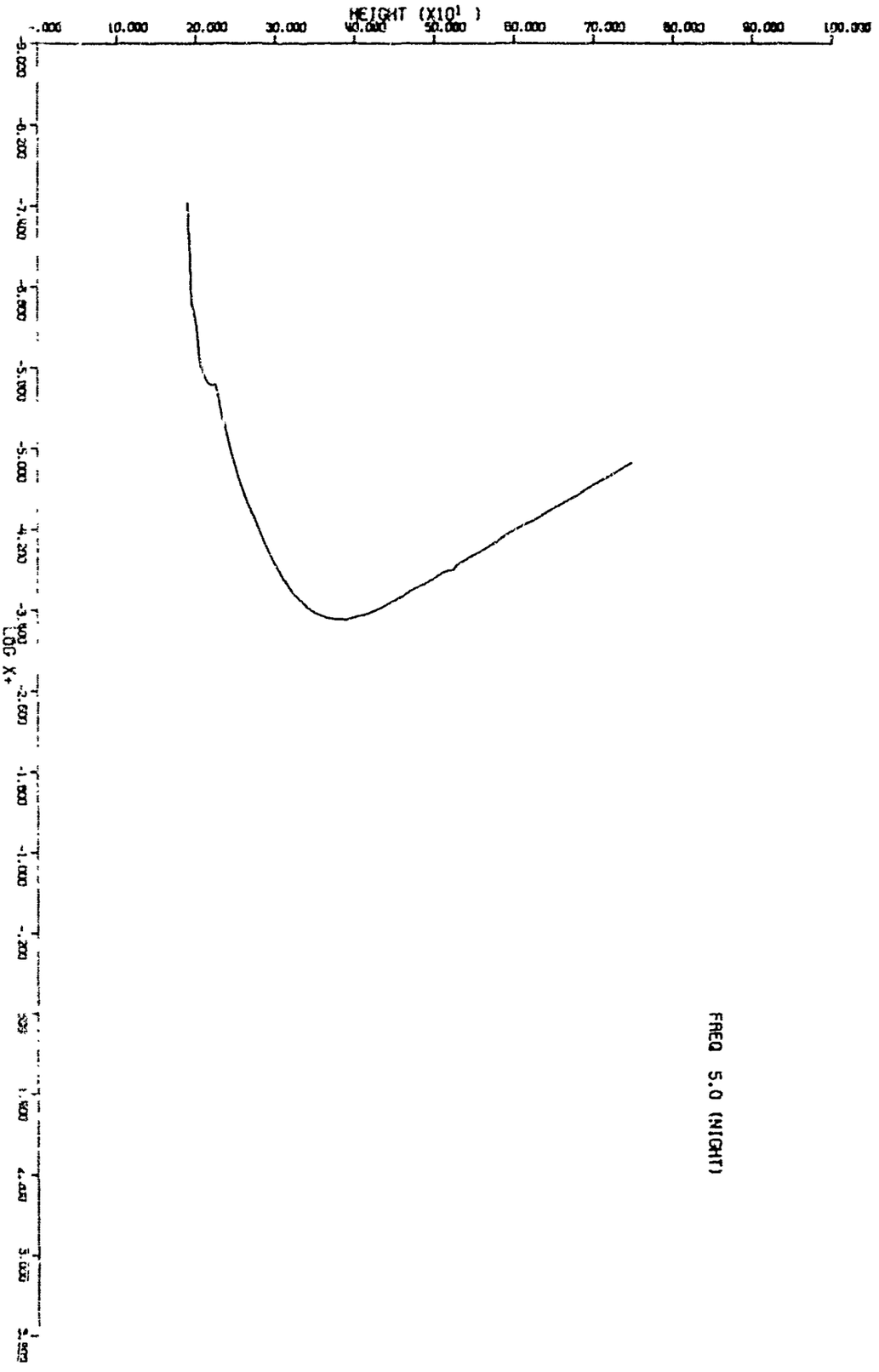
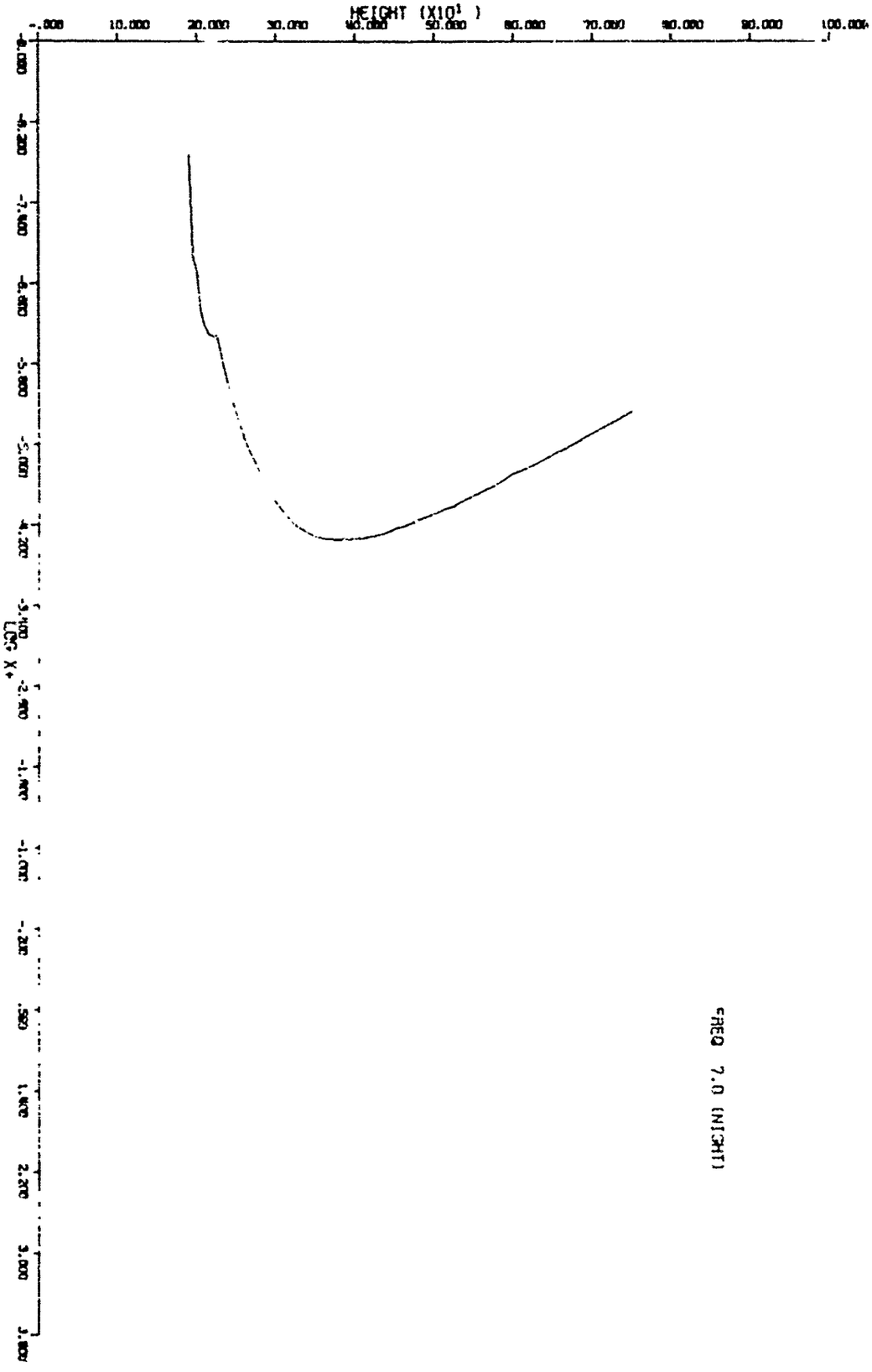
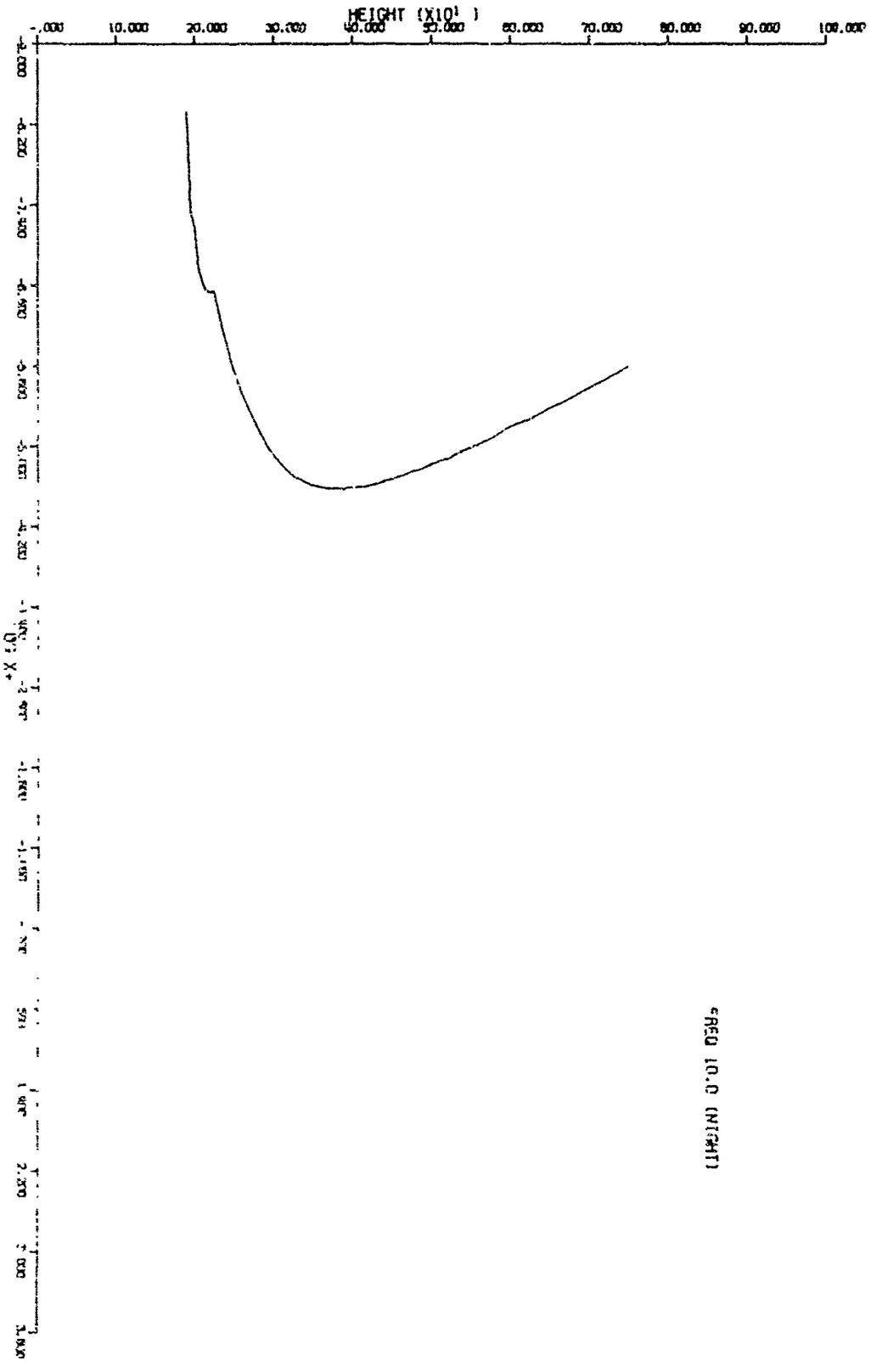


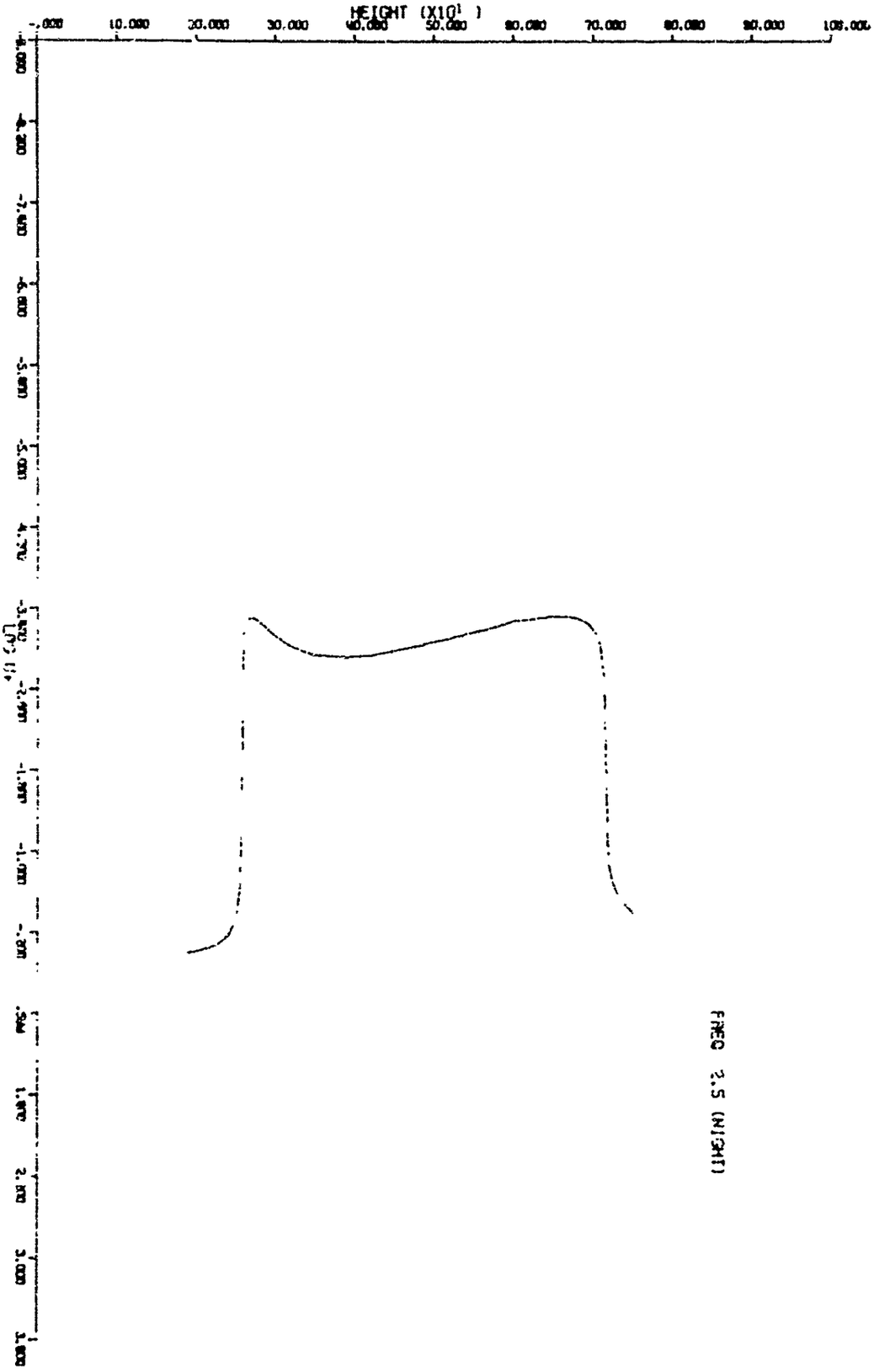
FIG 2.5 (HEIGHT)

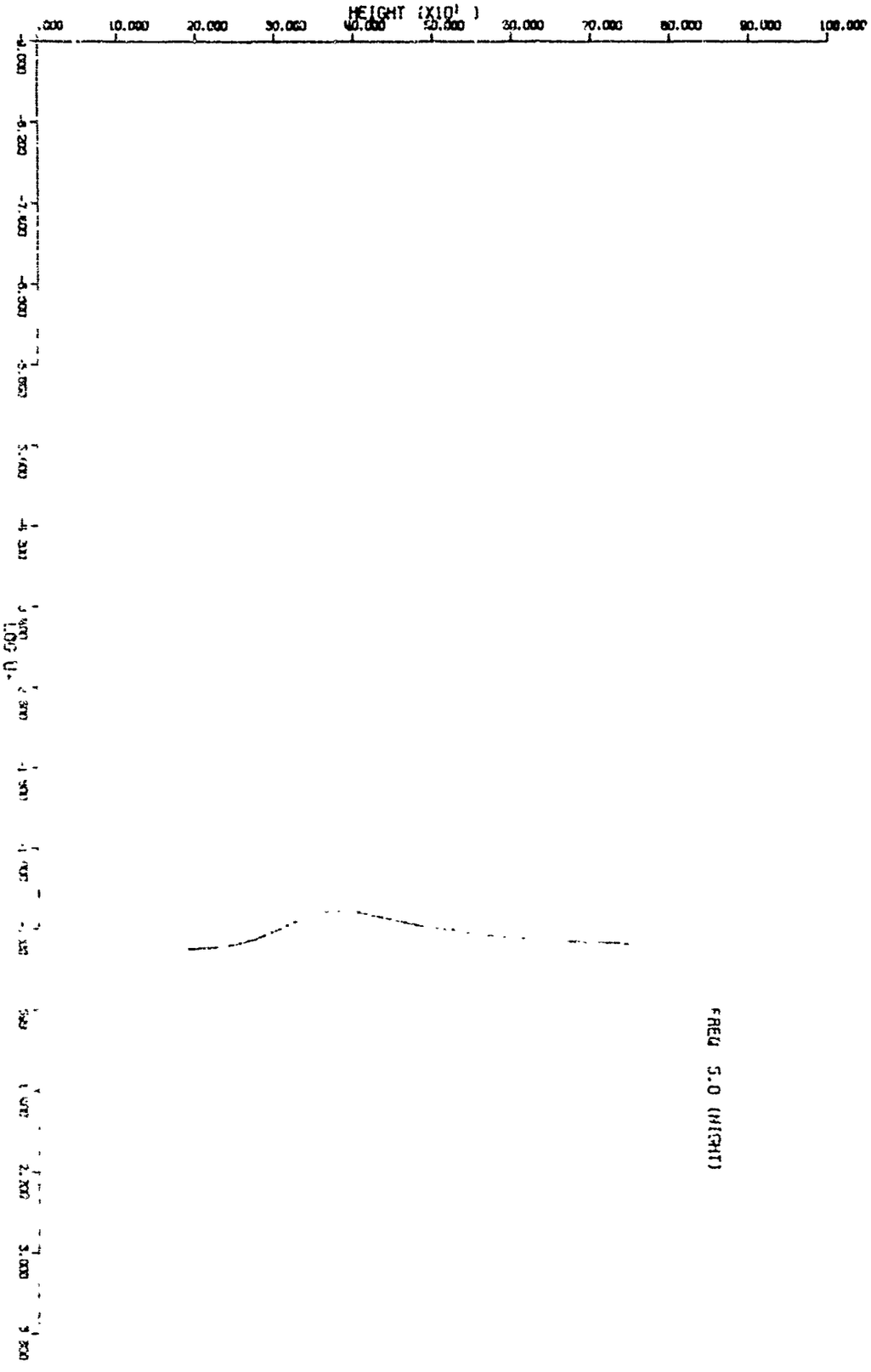


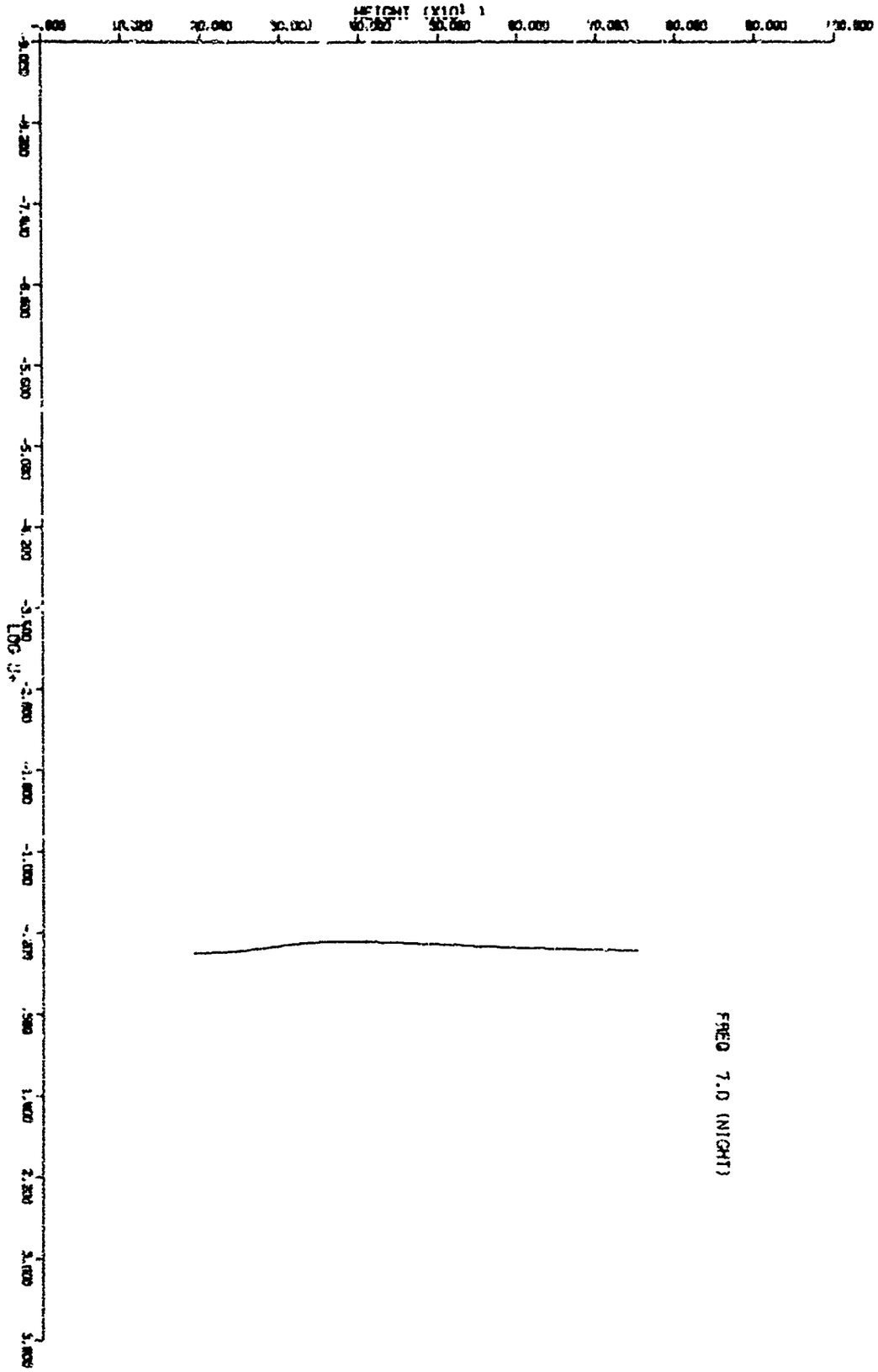
FREQ 5.0 (NIGHT)

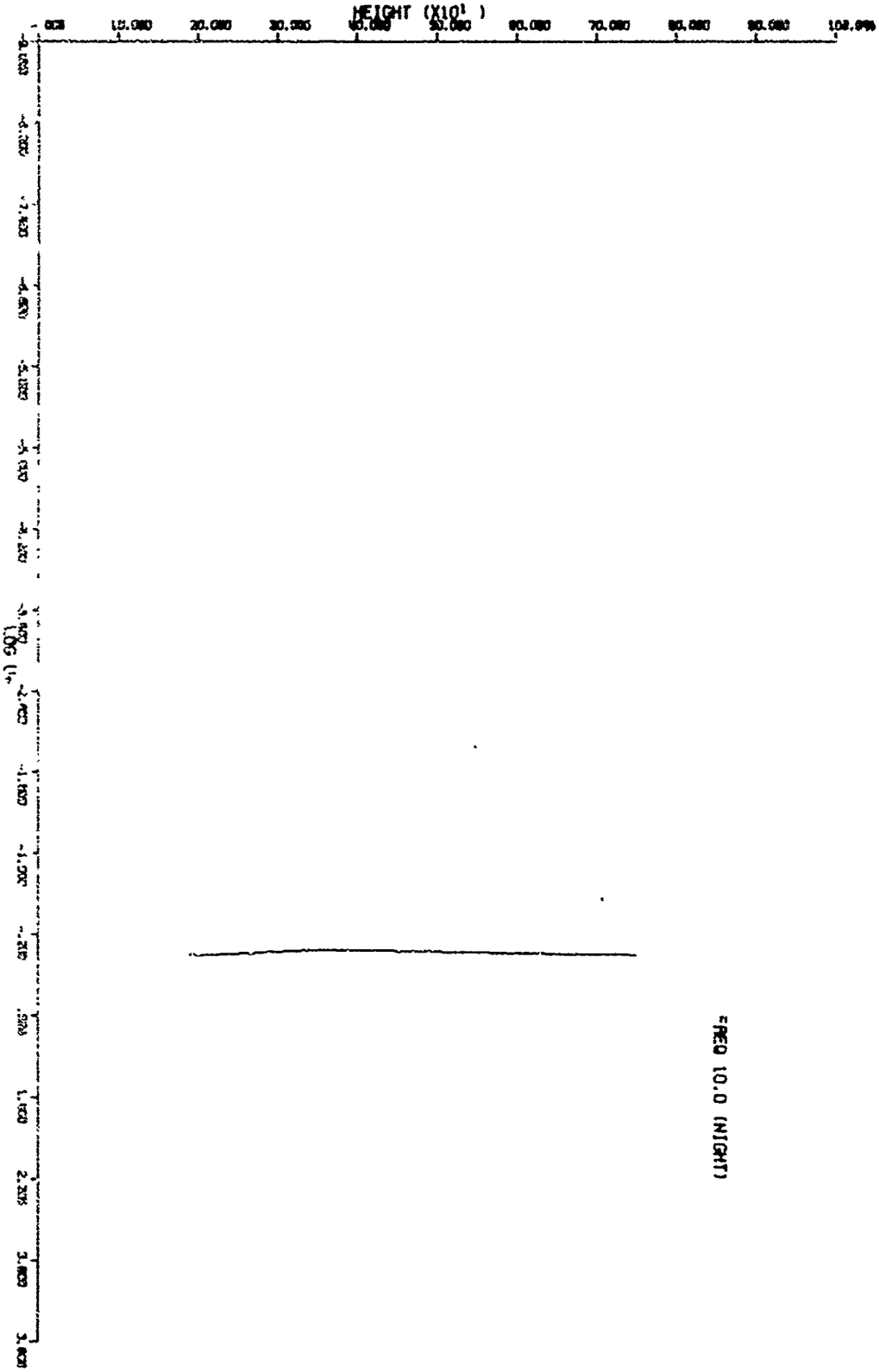

HEIGHT (X10¹)
10.000
20.000
30.000
40.000
50.000
60.000
70.000
80.000
90.000
100.000
LOG X+
FREQ 7.0 (NIGHT)


HEIGHT (X10¹)
-.000
10.000
20.000
30.000
40.000
50.000
60.000
70.000
80.000
90.000
100.000
AREO 10.0 (NIGHT)


HEIGHT (X10¹)
.000
10.000
20.000
30.000
40.000
50.000
60.000
70.000
80.000
90.000
100.000
FREQ 2.5 (MJ/KH1)


HEIGHT (X10³)
10.000
20.000
30.000
40.000
50.000
60.000
70.000
80.000
90.000
100.000
FREQ 5.0 (HIGH)

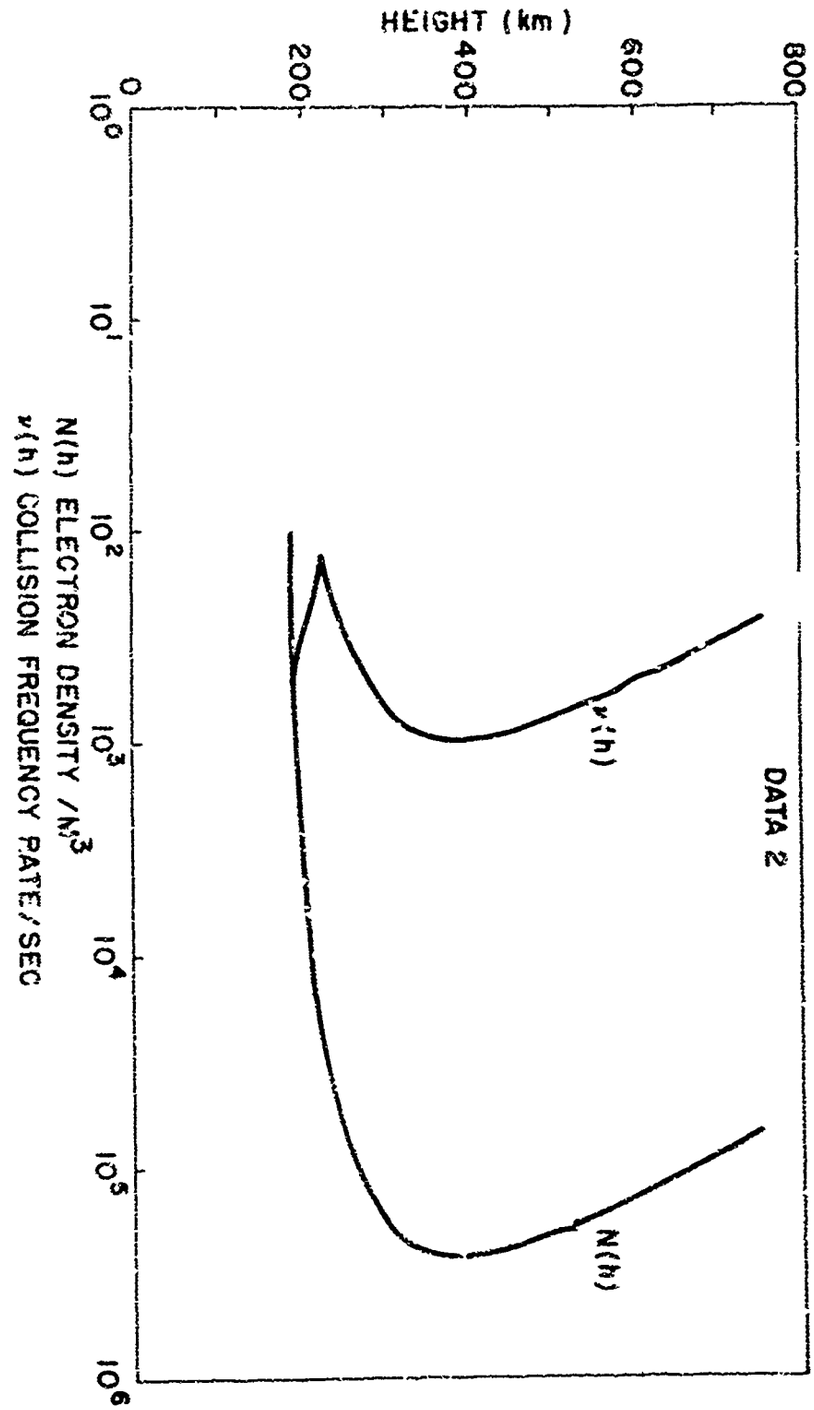

FREQ 7.0 (NIGHT)

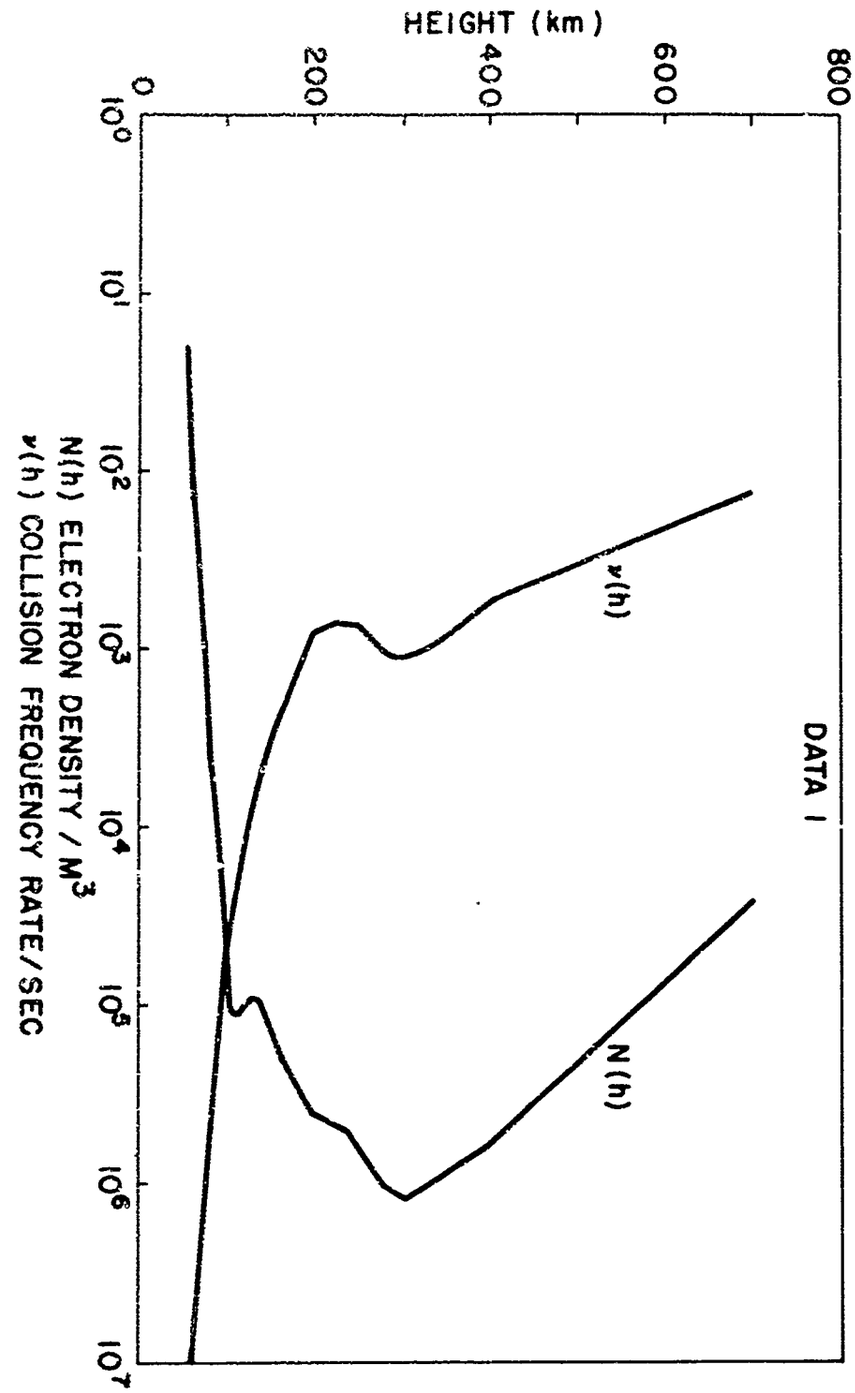

HEIGHT (X10¹)
LOG (1+
FREQ 10.0 (NIGHT)

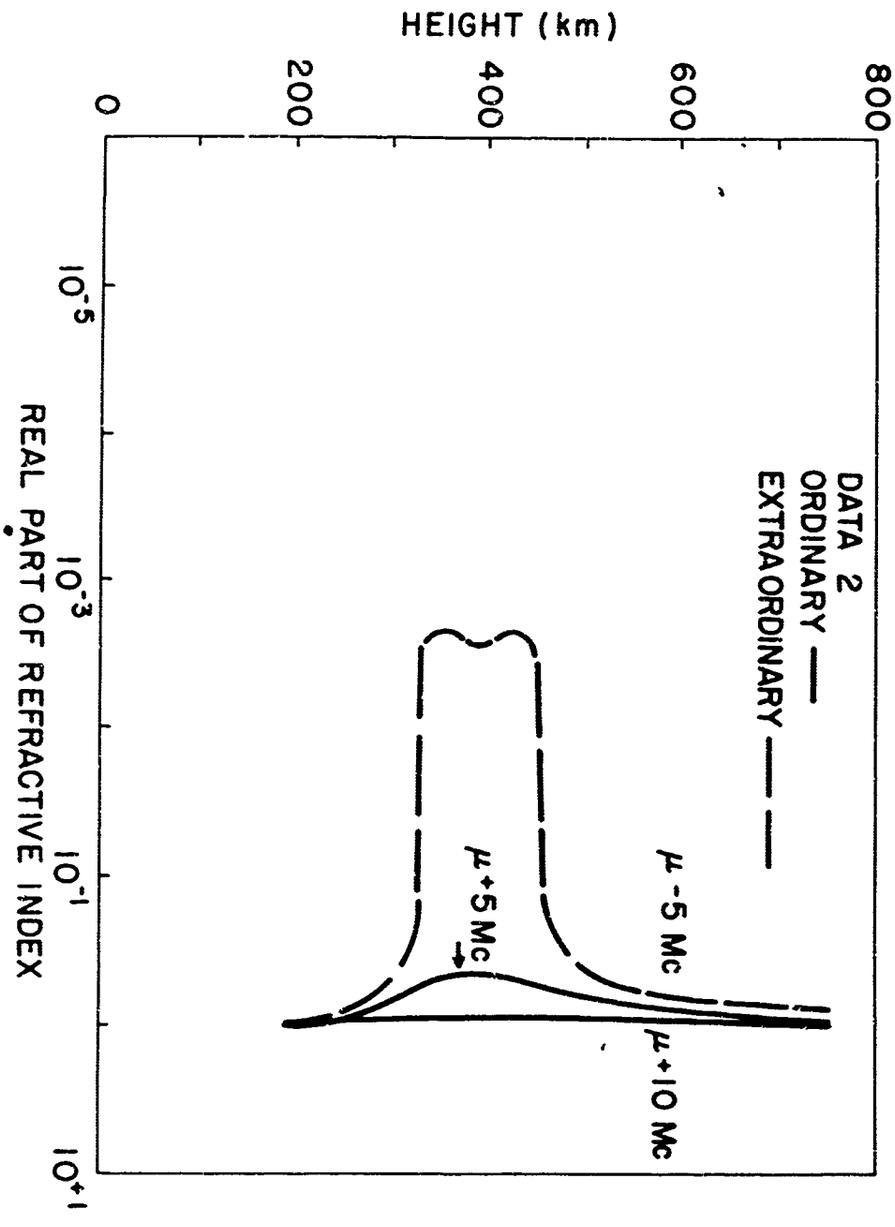
APPENDIX G

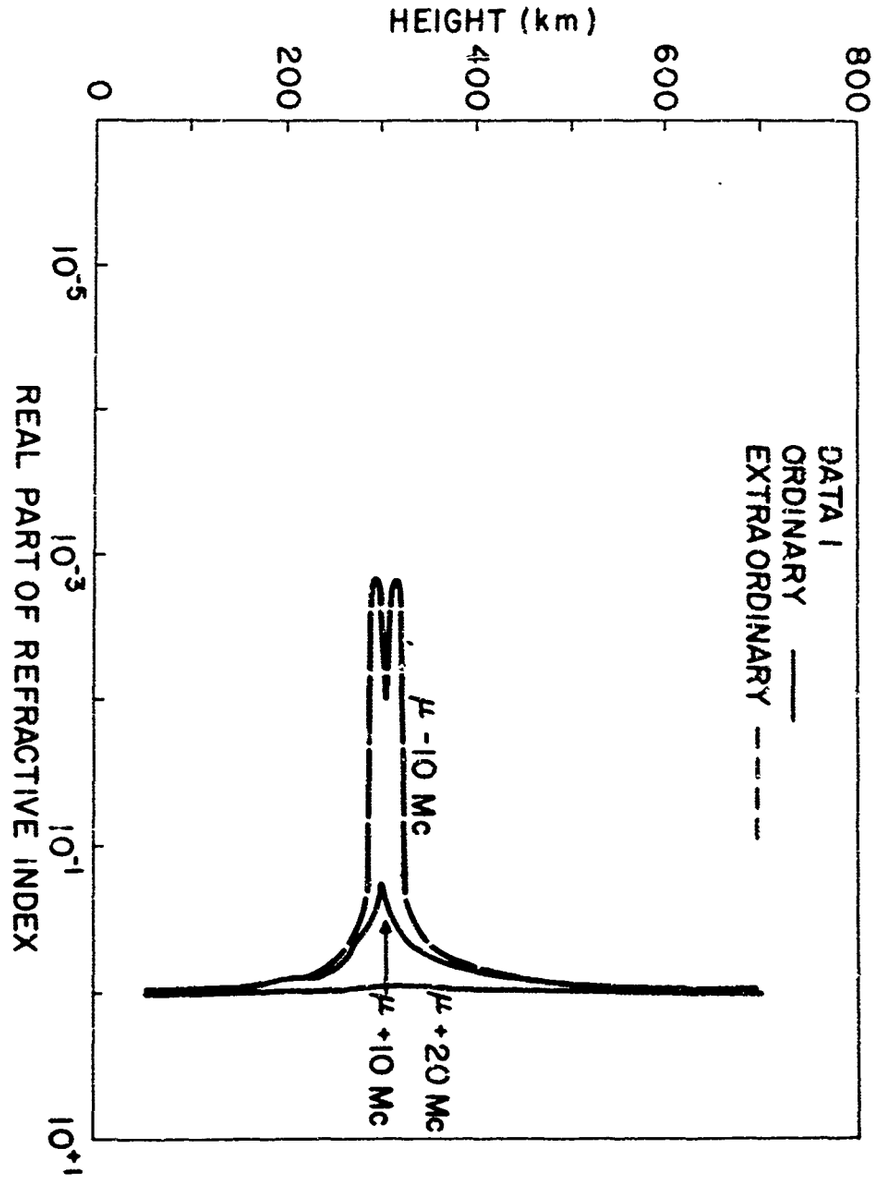
Grouped Height Profiles of U, χ

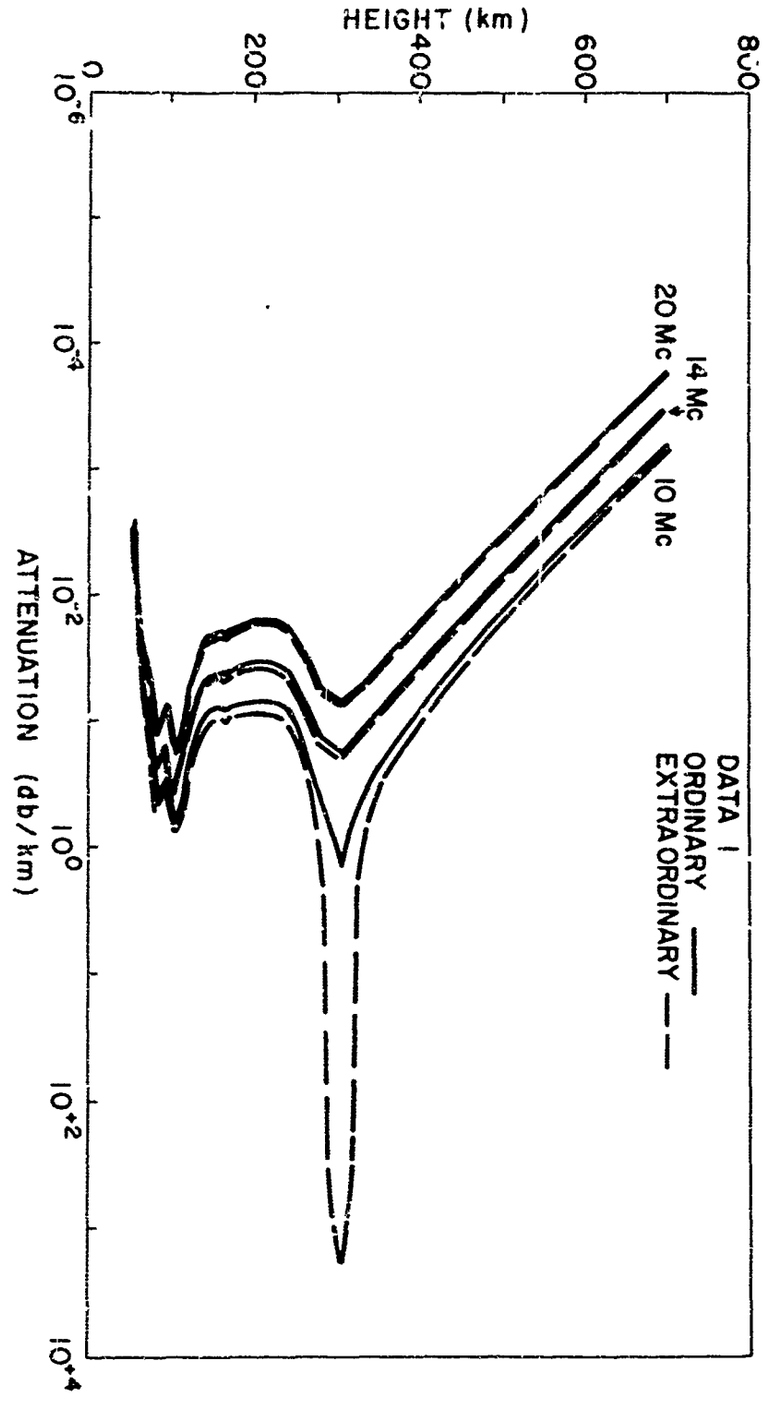
Part of the graphs presented in the previous appendix are presented but grouped together according to their being χ_o, χ_x, χ_o. Only those which were used in the determination of frequency dependence are plotted here. These are clearly self-explanatory. The grouping makes their frequency variation easier to see.

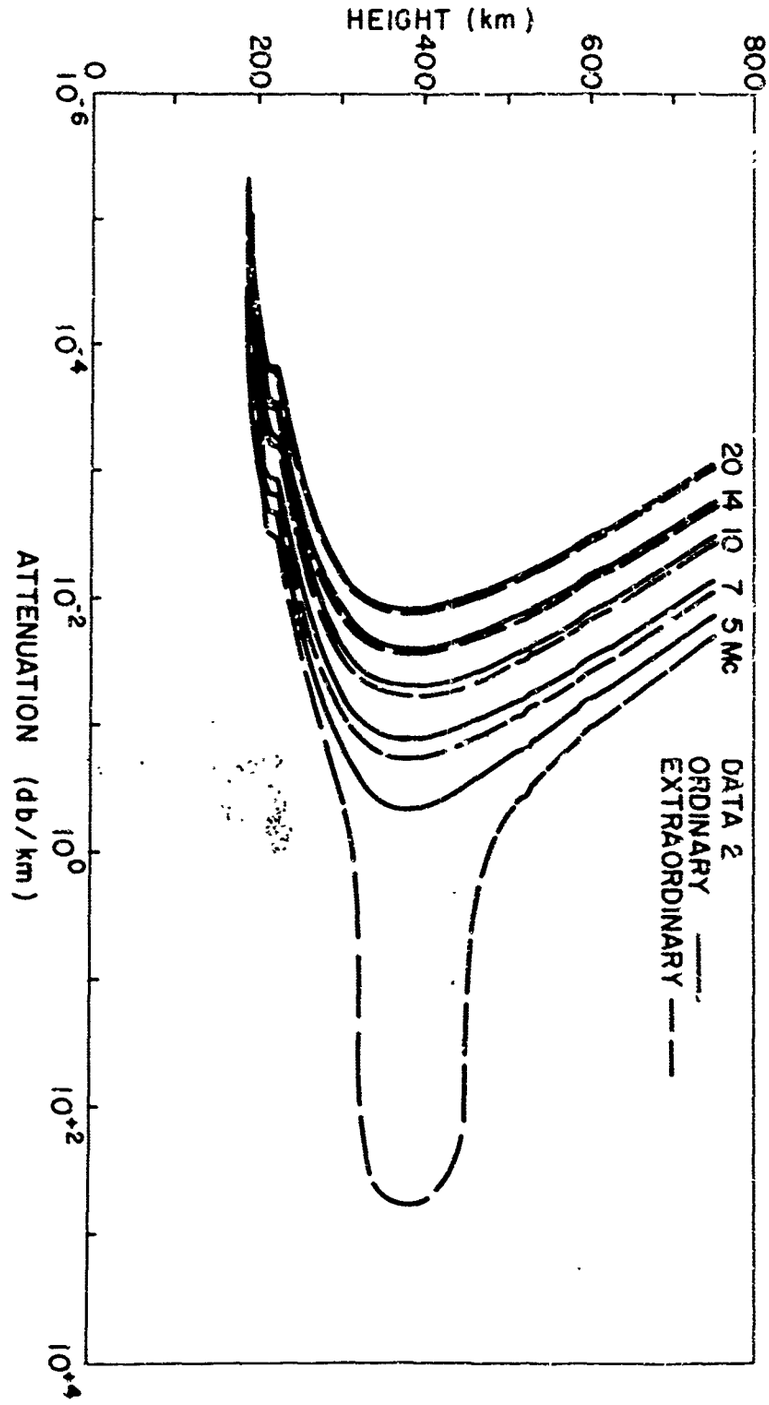

HEIGHT (km)
0
200
400
600
800
10^0
10^1
10^2
10^3
10^4
10^5
10^6
N(h) ELECTRON DENSITY /M^3
ν(h) COLLISION FREQUENCY RATE/SEC
ν(h)
N(h)
DATA 2


HEIGHT (km)
0
200
400
600
800
10^0
10^1
10^2
10^3
10^4
10^5
10^6
10^7
N(h) ELECTRON DENSITY / M^3
ν(h) COLLISION FREQUENCY RATE/SEC
ν(h)
N(h)
DATA I


HEIGHT (km)
0
200
400
600
800
10^{-5}
10^{-3}
10^{-1}
10^{+1}
REAL PART OF REFRACTIVE INDEX
DATA 2
ORDINARY ——
EXTRAORDINARY — — —
μ -5 Mc
μ +5 Mc
μ +10 Mc

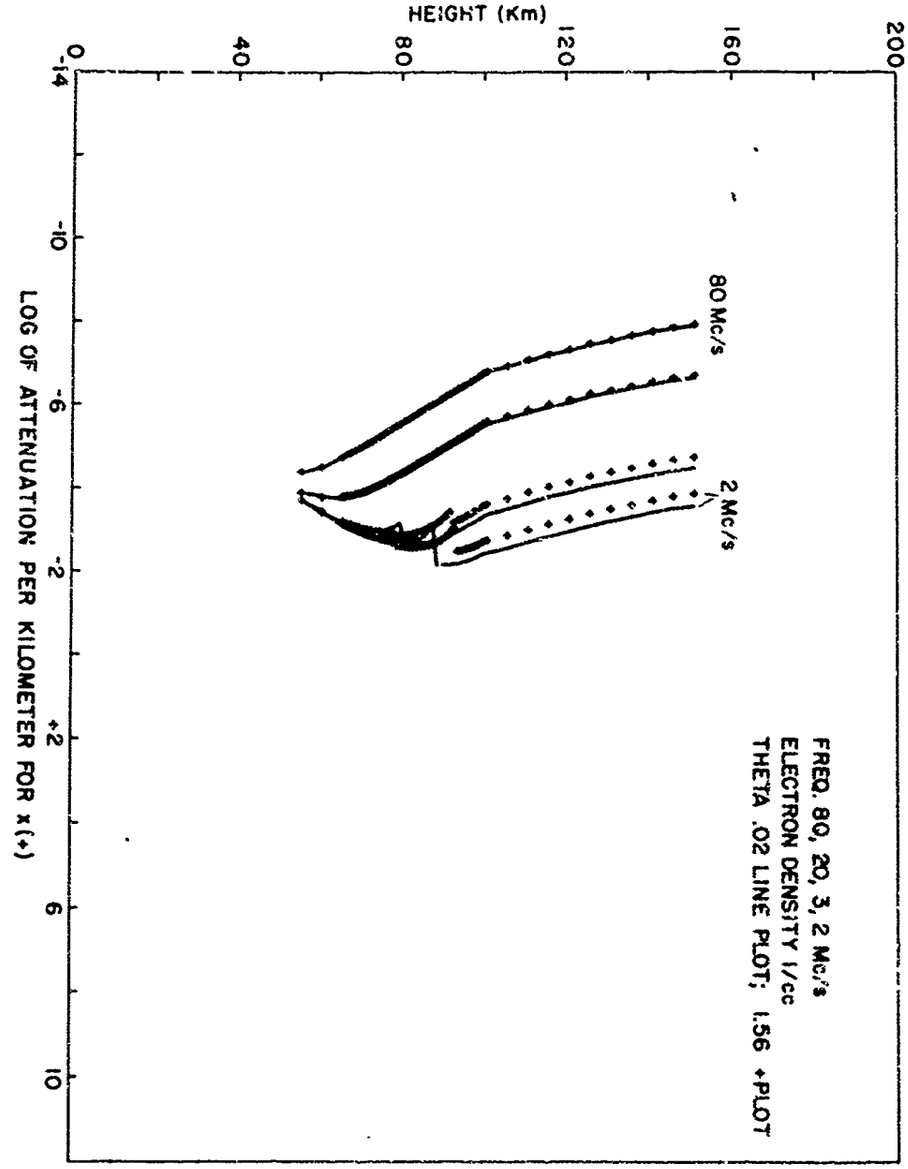

HEIGHT (km)
0
200
400
600
800
REAL PART OF REFRACTIVE INDEX
10^{-5}
10^{-3}
10^{-1}
10^{+1}
DATA I
ORDINARY ——
EXTRAORDINARY - - - -
μ -10 Mc
μ +10 Mc
μ +20 Mc

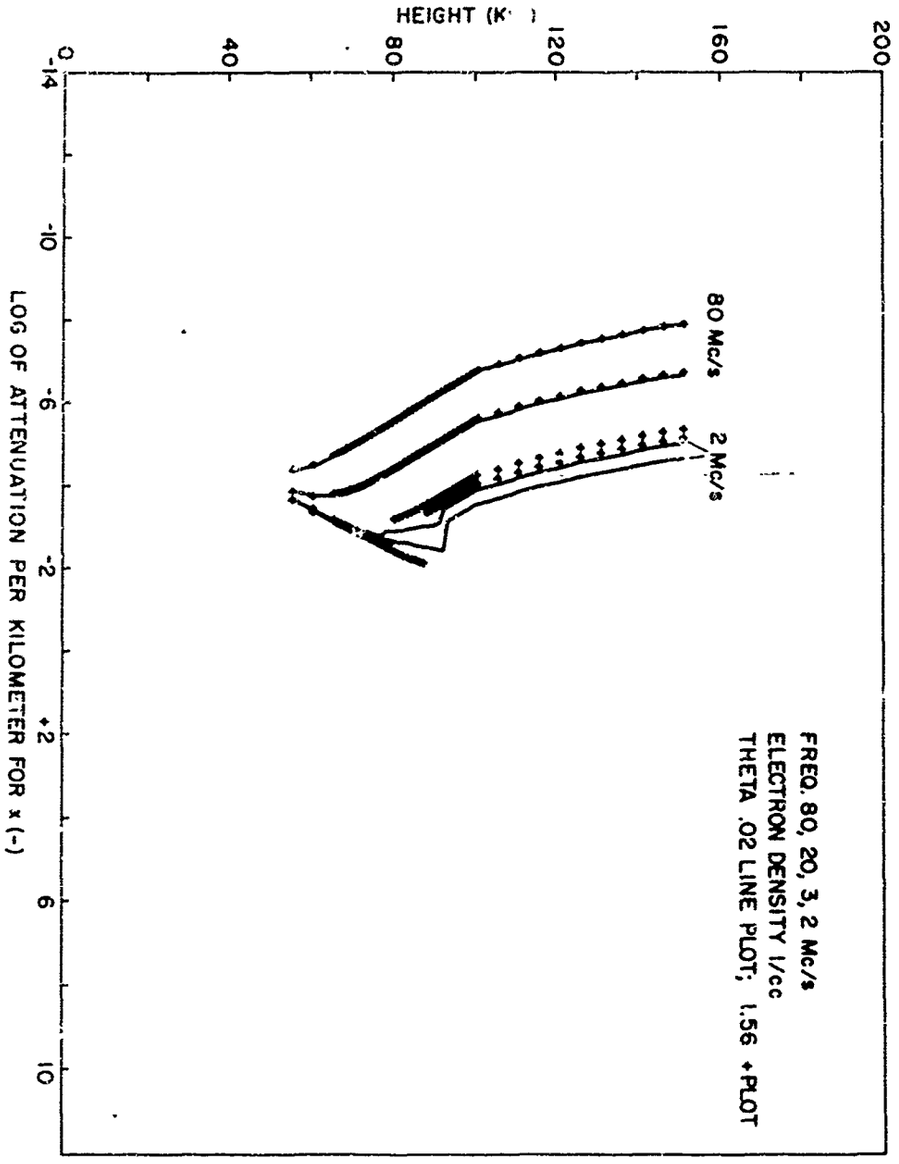

HEIGHT (km)
0
200
400
600
800
10^{-6}
10^{-4}
10^{-2}
10^{0}
10^{+2}
10^{+4}
ATTENUATION (db/km)
20 Mc
14 Mc
10 Mc
DATA I
ORDINARY
EXTRA ORDINARY

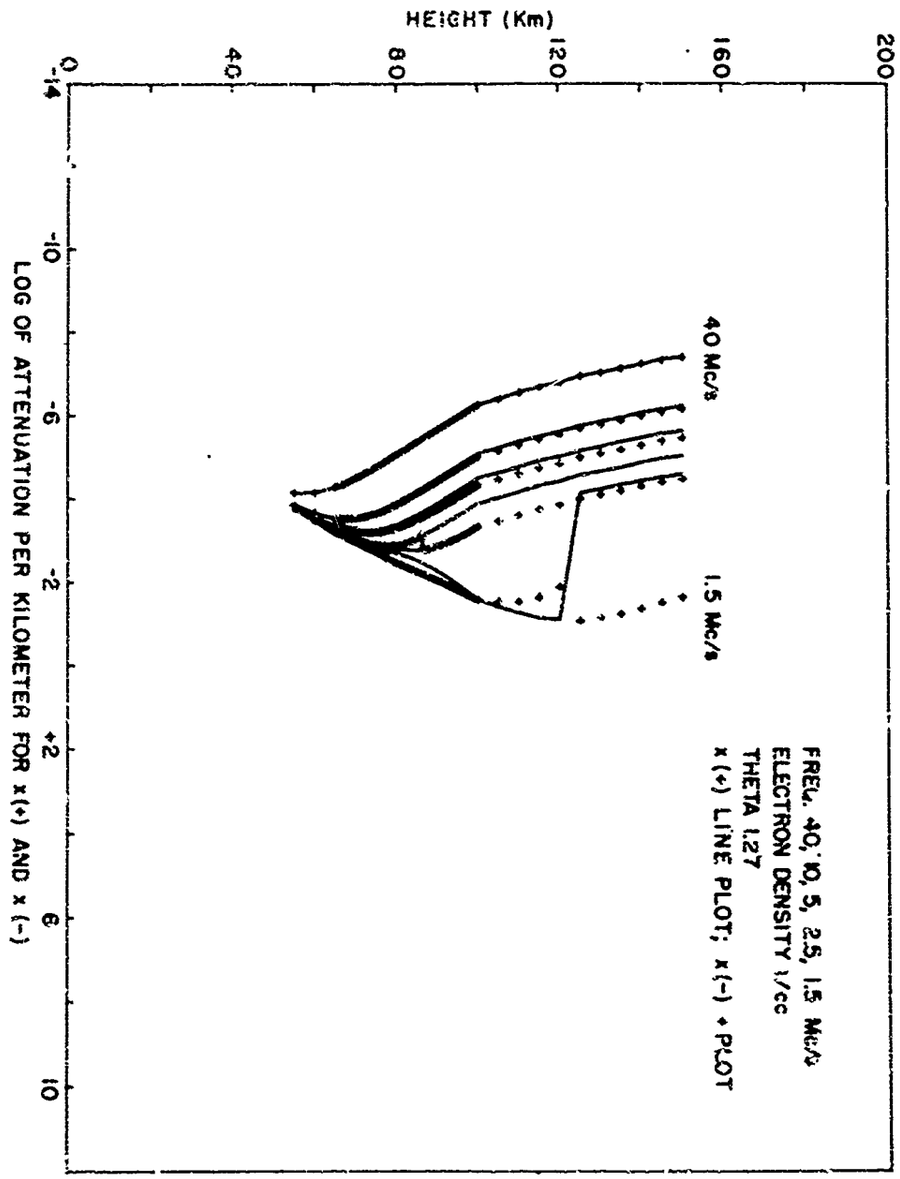

HEIGHT (km)
0
200
400
600
800
10^{-6}
10^{-4}
10^{-2}
10^{0}
10^{+2}
10^{+4}
ATTENUATION (db/km)
20 14 10 7 5 Mc
DATA 2
ORDINARY
EXTRAORDINARY

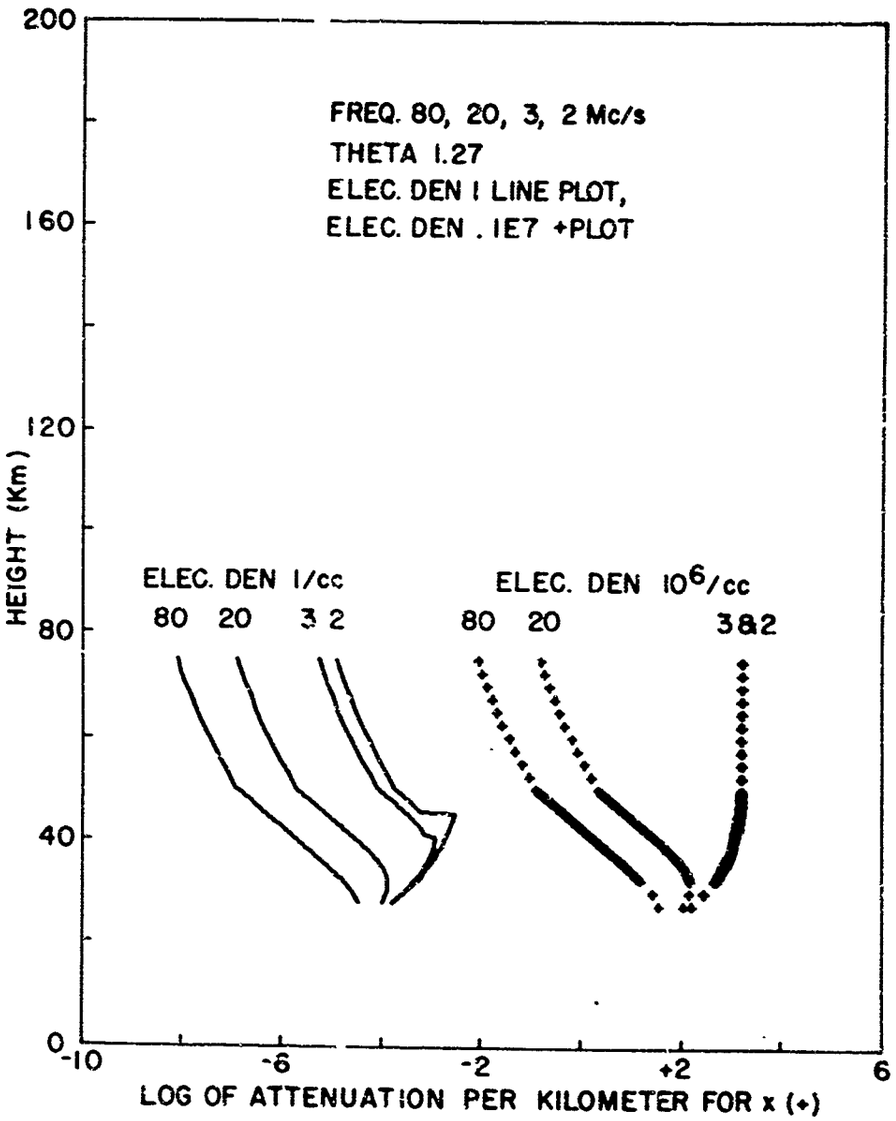
APPENDIX H

This appendix contains various graphs examining the relation between N and K_i i.e. under what conditions linearity holds. The frequencies used are stated in each of the graphs. The first two present $20(\log_{10}e)K_i$ for both the extraordinary and the ordinary ray against height. They are presented for both almost longitudinal conditions (THETA = .02 radians) and for almost transverse propagation (THETA = 1.56 radians). The third graph shows for both K_x, K_o, their variation with height when $\theta = 1.27$ and N = 1. The angle $\theta = 1.27$ was the one used in the general calculations presented before. The last one shows a comparison (for $\theta = 1.27$) between the attenuations given when N = 1 and when $N = 10^6/cc$.


HEIGHT (Km)
0
40
80
120
160
200
-14
-10
-6
-2
+2
6
10
LOG OF ATTENUATION PER KILOMETER FOR x(+)
80 Mc/s
2 Mc/s
FREQ. 80, 20, 3, 2 Mc/s
ELECTRON DENSITY 1/cc
THETA .02 LINE PLOT; 1.56 +PLOT


HEIGHT (K
0
40
80
120
160
200
-14
-10
-6
-2
+2
6
10
LOG OF ATTENUATION PER KILOMETER FOR x (-)
80 Mc/s
2 Mc/s
FREQ. 80, 20, 3, 2 Mc/s
ELECTRON DENSITY 1/cc
THETA .02 LINE PLOT; 1.56 +PLOT


HEIGHT (Km)
0
40
80
120
160
200
-14
-10
-6
-2
+2
6
10
LOG OF ATTENUATION PER KILOMETER FOR x(+) AND x(-)
40 Mc/s
1.5 Mc/s
FREQ. 40, 10, 5, 2.5, 1.5 Mc/s
ELECTRON DENSITY 1/cc
THETA 1.27
x(+) LINE PLOT; x(-) + PLOT


FREQ. 80, 20, 3, 2 Mc/s
THETA 1.27
ELEC. DEN 1 LINE PLOT,
ELEC. DEN . 1E7 +PLOT
ELEC. DEN 1/cc
80 20 3 2
ELEC. DEN 10^6/cc
80 20 3&2
HEIGHT (Km)
200
160
120
80
40
0
-10
-6
-2
+2
6
LOG OF ATTENUATION PER KILOMETER FOR x (+)